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Common fixed point results for noncommuting mappings without continuity in generalized metric spaces

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ABSTRACT

Using the setting of a generalized metric space, a fixed point theorem is proved for one map, and several fixed point theorems are proved for two maps. These results generalize several well known comparable results in the literature.

1. Introduction and preliminaries

The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity. In 1976, Jungck [3], proved a common fixed point theorem for commuting maps, generalizing the Banach contraction principle. This theorem has many applications but suffers from one drawback – the results require the continuity of one of the two maps involved. Kannan [14], proved the existence of a fixed point for a map that can have a discontinuity in its domain; however, the maps involved were continuous at the fixed point. Sessa [13] introduced the notion of weakly commuting maps. Jungck [4] coined the term compatible mappings in order to generalize the concept of weak commutativity and showed that weakly commuting maps are compatible but the converse is not true. Pant [11] defined R-weakly commuting maps and proved common fixed point theorems, assuming the continuity of at least one of the mappings. Jungck [6] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. In recent years, several authors have obtained coincidence point results for various classes of mappings on a metric space, utilizing these concepts. For a survey of coincidence point theory, its applications, comparison of different contractive conditions and related results, we refer to [1,5,7,11] and references contained therein. Mustafa and Sims [9] generalized the concept of a metric space. Recently, Mustafa et al. [10] obtained some fixed point theorems for mappings satisfying different contractive conditions. The first result of the paper is a coincidence point theorem for two maps satisfying a contractive condition in a generalized metric space. Common fixed point theorems for a pair of weakly compatible maps, which are more general than R-weakly commuting, and compatible mappings are obtained in the setting of a generalized metric spaces without exploiting the notion of continuity. Our results generalize Theorems 2.1–2.4 of [10]. Consistent with Mustafa and Sims [9], the following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies:

(a) $G(x,y,z) = 0$ if and only if $x = y = z$,
(b) $0 < G(x,y,z)$ for all $x,y \in X$, with $x \neq y$,
(c) $G(x,x,y) \leqslant G(x,y,z)$ for all $x,y,z \in X$, with $z \neq y$,

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(d) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables), and
(e) $G(x,y,z) \leqslant G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$.

Then G is called a G-metric on X and (X,G) is called a G-metric space.

Definition 1.2. A G-metric is said to be symmetric if $G(x,y,y) = G(y,x,x)$ for all $x,y \in X$.

Definition 1.3. Let (X,G) be a G-metric space. We say that $\{x_n\}$ is:

(f) a *G-Cauchy* sequence if, for any $\varepsilon > 0$, there is an $n_0 \in N$ (the set of all positive integers) such that for all $n,m,l \geqslant n_0$, $G(x_n,x_m,x_l) < \varepsilon$;
(g) a *G-convergent* sequence if, for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in N$, such that for all $n,m \geqslant n_0, G(x,x_n,x_m) < \varepsilon$.

A G-metric space X is said to be complete if every G-Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in (X,G)$ if and only if $G(x_m,x_n,x) \to 0$ as $n,m \to \infty$.

Definition 1.4. Let f and g be self maps of a set X. If $w = fx = gx$ for some x in X, then x is called a *coincidence point* of f and g, and w is called a *point of coincidence* of f and g.

Proposition 1.5. *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Proof. Since $w = fx = gx$ and f and g are weakly compatible, we have $fw = fgx = gfx = gw$: i.e., $fw = gw$ is a point of coincidence of f and g. But w is the only point of coincidence of f and g, so $w = fw = gw$. Moreover if $z = fz = gz$, then z is a point of coincidence of f and g, and therefore $z = w$ by uniqueness. Thus w is the unique common fixed point of f and g.

The following result is a consequence of Theorem 2.1 of [7]. □

Theorem 1.6. *Let C be a subset of a metric space (X,d), and f, and g be weakly compatible self maps of C. Assume that the range of g contains the range of $f, g(X)$ is a complete subspace of X, and f and g satisfy the condition*

$$d(fx,fy) \leqslant h\max\{d(gx,gy), d(gx,fy), d(gy,fx), d(gx,fx), d(gy,fy)\},$$

for all $x,y \in X$, where $0 \leqslant h < 1$. Then f and g have a unique common fixed point.

Let f be a self map of a space X. For $A \subseteq X$, set $\delta(A) = \sup\{d(x,y) : x,y \in A\}$. The set $O(x,n) = \{x, fx, \ldots, f^n x\}$. $O(x,\infty) = \{x, fx, f^2x, \ldots\}$ is called the orbit of x [2].

2. Common fixed point theorems

In this section we first obtain a fixed point theorem for a single map, and then obtain several coincidence and common fixed point theorems for mappings defined on a generalized metric space.

Theorem 2.1. *Let X be a complete G-metric space. Suppose that there exists a point $u \in X$ and a $\lambda \in [0,1)$ with $\overline{O(u)}$ complete and*

$$G(fx,fy,fy) \leqslant \lambda G(x,y,z), \tag{2.1}$$

for each $x, z = y = fx \in O(u)$. Then $\{f^n x\}$ converges to some point $p \in X$ and, for all $m,n \in \mathbb{N}, m > n$,

$$G(x_n,x_m,x_m) \leqslant \frac{\lambda^n}{1-\lambda} G(u,fu,fu),$$

for $n \geqslant 1$. Further, if f is orbitally continuous at p or if (2.1) holds for all $x \in \overline{O(u)}$, then p is a fixed point of f.

Proof. If G is symmetric, then

$$d_G(x,y) = 2G(x,y,y),$$

(2.1) becomes

$$d_G(fx,fy) \leqslant \lambda d_G(x,y),$$

for all $y \in O(x)$, and the result follows from Theorem 2 of [12]. Suppose that G is not symmetric. With $x_n := f^n u$, one has, from (2.1), that

$$G(x_{n+1},x_{n+2},x_{n+2}) \leqslant \lambda G(x_n,x_{n+1},x_{n+1}) \leqslant \cdots \leqslant \lambda^n G(u,fu,fu).$$