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Oscillation criteria for a forced mixed type Emden–Fowler equation with impulses ☆

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ABSTRACT

Some oscillation criteria for a forced mixed type Emden–Fowler equation with impulses are given. When the impulses are dropped, our results extend those of Sun and Meng [Y.G. Sun, F.W. Meng, Interval criteria for oscillation of second-order differential equations with mixed nonlinearities, Appl. Math. Comput. 15 (2008) 375–381], Sun and Wong [Y.G. Sun, J.S.W. Wong, Oscillation criteria for second order forced ordinary differential equations with mixed nonlinearities, J. Math. Anal. Appl. 334 (2007) 549–560] for second-order forced ordinary differential equation with mixed nonlinearities, Nasr [A.H. Nasr, Sufficient conditions for the oscillation of forced superlinear second order differential equations with oscillatory potential, Proc. Am. Math. Soc. 126 (1998) 123–125], Yang [Q. Yang, Interval oscillation criteria for a forced second order nonlinear ordinary differential equations with oscillatory potential, Appl. Math. Comput. 135 (2003) 49–64] for forced superlinear Emden–Fowler equation, Kong [Q. Kong, Interval criteria for oscillation of second-order linear differential equations, J. Math. Anal. Appl. 229 (1999) 483–492] for unforced second order linear differential equations, and Wong [J.S.W. Wong, Oscillation criteria for a forced second order linear differential equation, J. Math. Anal. Appl. 231 (1999) 235–240] for forced second order linear differential equation.

1. Introduction

Consider the forced mixed type Emden–Fowler equation with impulses

$$\begin{cases} (r(t)x'(t))' + p(t)x(t) + \sum_{i=1}^{n} p_i(t)|x(t)|^{\alpha_i-1}x(t) = e(t), & t \neq \tau_k, \\ x(\tau_k^+) = a_k x(\tau_k),\ x'(\tau_k^+) = b_k x'(\tau_k), \end{cases} \tag{1.1}$$

where $t \geqslant t_0$, $k \in \mathbb{N}$, $\{\tau_k\}$ is the impulse moments sequence with $0 \leqslant t_0 = \tau_0 < \tau_1 < \cdots < \tau_k < \cdots$, $\lim_{k\to\infty}\tau_k = +\infty$, and

$$x(\tau_k) = x(\tau_k^-) = \lim_{t\to\tau_k-0} x(t), \quad x(\tau_k^+) = \lim_{t\to\tau_k+0} x(t),$$

$$x'(\tau_k) = x'(\tau_k^-) = \lim_{h\to 0-} \frac{x(\tau_k+h) - x(\tau_k)}{h}, \quad x'(\tau_k^+) = \lim_{h\to 0+} \frac{x(\tau_k+h) - x(\tau_k^+)}{h}.$$

Here, we always assume that the following conditions hold:

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(H1) $r \in C^1([t_0,\infty),(0,\infty)), p, p_i, e \in C([t_0,\infty),\mathbb{R}), i = 1,\dots,n$;
(H2) $\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$ are constants;
(H3) $b_k \geqslant a_k > 0, k \in \mathbb{N}$, are constants.

We note that the impulsive differential equations are an adequate mathematical apparatus for simulation of process and phenomena observed on control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, economics, etc. For further applications and questions concerning existence and uniqueness of solutions of impulsive differential equations, see [5].

Let $J \subset \mathbb{R}$ be an interval, we define

$$PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x(t) \text{ is piecewise-left-continuous and has discontinuity of first kind at } \tau_k' s.\}$$

By a solution of Eq. (1.1), we mean a function $x \in PC([t_0,\infty),\mathbb{R})$ with the property $(rx')' \in PC([t_0,\infty),\mathbb{R})$ such that (1.1) is fulfilled for all $t \geqslant t_0$, and $\sup\{|x(t)| : t \geqslant t_x\} > 0$ for some $t_x \geqslant t_0$. It is tacitly assumed that such solutions exist. As usual, a solution of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise, it is called nonoscillatory. The equation is called oscillatory if every solution is oscillatory.

When the impulses are dropped, Eq. (1.1) reduces to the second-order forced ordinary differential equation with mixed nonlinearities

$$(r(t)x'(t))' + p(t)x(t) + \sum_{i=1}^{n} p_i(t)|x(t)|^{\alpha_i - 1}x(t) = e(t), \tag{1.2}$$

which can be considered the natural generalizations of the linear nonhomogeneous equation

$$(r(t)x'(t))' + p(t)x(t) = e(t), \tag{1.3}$$

and the forced Emden–Fowler equation

$$x''(t) + p(t)|x(t)|^{\lambda-1}x(t) = e(t). \tag{1.4}$$

If $\lambda > 1$, Eq. (1.4) is known as the superlinear equation and if $0 < \lambda < 1$, it is known as the sublinear equation. The oscillation of Eqs. (1.2) and (1.3) has been the subject of much attention during the last 50 years, see, for example, [1] and the vast references cites therein. Especially, in 1993, by using a Leighton comparison type argument [6], El-Sayed [2] removed the sign condition $p(t) \geqslant 0$ previously imposed, and obtained sufficient conditions for oscillation of (1.3) with $r(t) \equiv 1$, see also Wong [15] for a similar result. The oscillation of Eq. (1.4) can be found in Nasr [10], see also [3,7,8,14,16] for some related works about Eq. (1.4).

As we known, almost all existing oscillation criteria in the literature, see for example [1–3,7,8,10,14–16], are established only for the linear Eq. (1.3) or the superliner equation (1.4). However, for mixed type Emden–Fowler equation (i.e., the equation contains a finite sum of powers of x and if there exists in sum exponents of which are both greater than and less than 1), which arise, for instant, in the growth of bacteria population with competitive species, the study for the oscillation for this mixed type Emden–Fowler equation is very scare. To the best of our knowledge, the first paper in this direction is [13], in which Nasr–Wong oscillation criteria [10,15] are establish for Eq. (1.2), see also [12].

Our purpose here is to establish Sun–Wong type theorems for Eq. (1.1). Motivated by the ideas in [9,10,13,15], in this paper, we employ the arithmetic–geometric mean inequality [4] to obtain oscillation criteria for Eq. (1.1). When the impulses are dropped, our results extend those of Sun and Meng [12], Sun and Wong [13] for Eq. (1.2), Nasr [10] and Yang [16] for Eq. (1.4), Kong [9] for Eq. (1.3), Wong [15] for Eq. (1.3). Finally, we give two examples to illustrate our main results.

2. Main results

For convenience, we introduce the following notations. Let

$$k(s) = \max\{i : t_0 < \tau_i < s\}$$

and for $c_j < d_j$, let $r_j = \max\{r(t) : t \in [c_j, d_j]\}$, and

$$\Omega(c_j, d_j) = \{w \in C^1([c_j, d_j],\mathbb{R}) : w(t) \not\equiv 0, w(c_j) = w(d_j) = 0\}, \quad j = 1, 2.$$

For two constants $c, d \notin \{\tau_k\}$ with $c < d$, and a function $\phi \in C([c,d],\mathbb{R})$, we define an operator $Q : C([c,d],\mathbb{R}) \to \mathbb{R}$ by

$$Q_c^d[\phi] = \begin{cases} 0, & \text{for } k(c) = k(d), \\ \phi(\tau_{k(c)+1})\theta(c) + \sum\limits_{i=k(c)+2}^{k(d)} \phi(\tau_i)\varepsilon(\tau_i), & \text{for } k(c) < k(d), \end{cases}$$

where

$$\theta(c) = \frac{b_{k(c)+1} - a_{k(c)+1}}{a_{k(c)+1}(\tau_{k(c)+1} - c)}, \quad \varepsilon(\tau_i) = \frac{b_i - a_i}{a_i(\tau_i - \tau_{i-1})}.$$

To prove our main results, we need the following lemma due to Sun and Wong [13].