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On the uniform convergence of interpolating polynomials

James Guyker

Department of Mathematics, Buffalo State College, 1300 Elmwood Avenue, Buffalo, NY 14222-1095, United States

Abstract

A sufficient condition is given for a continuous power series $f(x)$ on [0, 1] to be the uniform limit of its sequence P_nf of interpolating polynomials at $n+1$ equally spaced nodes. The proof is based on expanding the Newton coefficients of P_nf in terms of Stirling numbers of the second kind and applying an Abel-like summation formula. Convergence rates of P_nf and of related coefficient sequences are estimated. Similar results follow for Bernstein polynomials and their derivatives.

Keywords: Power series; Interpolating polynomial; Uniform convergence; Stirling number of the second kind; Bernstein polynomial

1. Introduction

Let f be in $C[a,b]$ and let P_nf denote the unique polynomial of degree at most n which interpolates f at $n+1$ equally spaced nodes. *When does the sequence P_nf converge uniformly to f?* [1–4] The classical example $f(x) = \frac{1}{1+x^2}$ on $[-5,5]$ of C. Runge satisfies

$$\lim_{n\to\infty} \|f - P_nf\|_\infty = \infty.$$

Moreover, S. Bernstein showed that if $f(x) = |x|$ on $[-1,1]$, then $(P_nf)(x)$ converges to $f(x)$ only for $x = 0, \pm 1$. However, a positive answer to this question based on Hermite's theorem [2,3] is possible under the assumption that there exists an analytic continuation of f to a certain type of domain in the complex plane containing $[a,b]$. In this note we obtain an affirmative answer by elementary real methods for functions f defined by power series, and estimate the rates of convergence of P_nf and of related coefficient sequences. These ideas are then applied to the uniform convergence of Bernstein polynomials and their derivatives.

We follow the custom of assuming that the underlying interval is $[0,1]$ since if f is in $C[a,b]$ then $g(x) := f(a+(b-a)x)$ is in $C[0,1]$ and $f(x) = g\left(\frac{x-a}{b-a}\right)$. A key to our arguments is an analog of the Abel partial summation formula [5]:

$$\sum_{i=m}^{n} a_ib_i = \left(\sum_{i=m}^{n} a_i\right)b_m + \sum_{j=m}^{n-1}\left(\sum_{i=j+1}^{n} a_i\right)(b_{j+1} - b_j). \tag{1}$$

E-mail address: guykerj@buffalostate.edu

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2. Interpolating polynomials for power series

By Abel's theorem [6] a power series $\sum a_i x^i$ is in $C[0,1]$ if and only if the series $\sum a_i$ converges.

Theorem 1. *Let $f(x)$ be defined on $[0,1]$ by the power series $\sum a_i x^i$ where $\sum a_i$ converges, and let $P_n f$ be the polynomial of degree at most n which interpolates f at $n+1$ equally spaced nodes.*

Then

$$\|f - P_n f\|_\infty \leqslant 2[1 + (1+e^{-1})^{n-1}]\epsilon_n(\langle a_i \rangle)$$

where

$$\epsilon_n(\langle a_i \rangle) := \max\left\{ |\sum_{i>k} a_i| : k \geqslant n \right\}$$

converges to zero.

Proof. Let $f(x) = \sum a_i x^i$ where $\sum a_i$ converges. The Newton form of $P_n f$ simplifies for equally spaced nodes [1,2]:

$$(P_n f)(x) = a_0 + \sum_{j=1}^{n} c_{nj} x \left(x - \frac{1}{n} \right) \cdots \left(x - \frac{j-1}{n} \right)$$

with

$$c_{nj} := \frac{n^j}{j!} \Delta^j f(0), \tag{2}$$

where

$$\Delta^0 f(x) := f(x) \quad \text{and} \quad \Delta^{j+1} f(x) := \Delta^j f\left(x + \frac{1}{n} \right) - \Delta^j f(x).$$

The unsigned coefficients of the forward differences $\Delta^j f(0)$ may be obtained from the jth row of Pascal's triangle:

$$\Delta^j f(0) = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} f\left(\frac{k}{n} \right).$$

Thus since $\sum a_i$ converges,

$$\Delta^j f(0) = j! \sum_i S(i,j) \frac{a_i}{n^i},$$

where

$$S(i,j) = \frac{1}{j!} \sum_{k=0}^{j} (-1)^{j-k} k^i \binom{j}{k} \tag{3}$$

is a Stirling number of the second kind [7,8] and is given by the recurrence relation

$$S(i,j) = S(i-1, j-1) + jS(i-1, j) \tag{4}$$

with

$$S(i,1) = S(i,i) = 1 \quad \text{and} \quad S(i,j) = 0 \quad \text{for } j > i.$$

Therefore

$$c_{nj} = \sum_i \frac{S(i,j)}{n^{i-j}} a_i. \tag{5}$$