Applied Mathematics and Computation 216 (2010) 1226–1234

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Bifurcation and chaos in an epidemic model with nonlinear incidence rates

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ARTICLE INFO

Keywords:
Epidemic model
Discrete-time
Nonlinear incidence rates
Transcritical bifurcation
Flip bifurcation
Hopf bifurcation
Chaos
Rich dynamics

ABSTRACT

This paper investigates a discrete-time epidemic model by qualitative analysis and numerical simulation. It is verified that there are phenomena of the transcritical bifurcation, flip bifurcation, Hopf bifurcation types and chaos. Also the largest Lyapunov exponents are numerically computed to confirm further the complexity of these dynamic behaviors. The obtained results show that discrete epidemic model can have rich dynamical behavior.

1. Introduction

Mathematical models describing the population dynamics of infectious diseases have been playing an important role in better understanding of epidemiological patterns and disease control for a long time. Epidemiological models are now widely used as more epidemiologists realize the role that modeling can play in basic understanding and policy development [10,19]. Understanding emergent infectious diseases in humans is viewed with increasing importance. The rapid spread of SARS [4], the perceived threat of bio-terrorism [15] and concerns over influenza pandemics [23] have all highlighted vulnerability to (re)emerging infections. For all these examples, mathematical modeling has been used to develop an understanding of the relevant epidemiology, as well as to quantify the likely effects of different intervention strategies [8,11,21].

An important aspect of the mathematical study of epidemiology is the formulation of the incidence function. The incidence rate is the rate of new infection. In most epidemiological models, bilinear and standard incidence rates have been frequently used in classical epidemic models [2,3,5,7,12–14,18]. Liu et al. [16,17] concluded that the bilinear mass action incidence rate due to saturation or multiple exposures before infection could lead to nonlinear incidence rate $\beta S^p I^q$.

Simple models, by their own nature, cannot incorporate many of the complex biological factors. However, they often provide useful insights to help our understanding of complex process. For such reason, in the present study, we set $p = 2$ and $q = 1$. We firstly focus on the following continuous model:

$$\begin{cases} \frac{dS}{d\tau} = rS(1 - \frac{S}{K}) - \beta S^2 I, \\ \frac{dI}{d\tau} = \beta S^2 I - dI, \end{cases} \tag{1.1}$$

where S, I are denoted as the susceptible and infected, respectively. And r represents the intrinsic birth rate constant, K represents the carrying capacity, β represents the force of infection or the rate of transmission, and d represents the death coefficient of I for the disease.

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doi:10.1016/j.amc.2010.02.014

By setting

$$\tau = \frac{t}{d}, \quad b = \frac{\beta}{d}, \quad a = \frac{r}{d},$$

we have the following form:

$$\begin{cases} \frac{dS}{dt} = aS(1 - \frac{S}{K}) - bS^2I, \\ \frac{dI}{dt} = bS^2I - I. \end{cases} \tag{1.2}$$

The advantages of a discrete-time approach are multiple in epidemic model [20,22]. Firstly, difference models are more realistic than differential ones since the epidemic statistics are compiled from given time intervals and not continuously. The second reason are that, the discrete-time models can provide natural simulators for the continuous cases. One can thus not only study with good accuracy the behavior of the continuous-time model, but also assess the effect of larger time steps. At last, the use of discrete-time models makes it possible to use the entire arsenal of methods recently developed for the study of mappings and lattice equations, either from the integrability and/or chaos points of view.

Applying Euler scheme to the system (1.2), we obtain the following equation:

$$\begin{cases} S_{n+1} = (a+1)S_n - cS_n^2 - bS_n^2 I_n, \\ I_{n+1} = bS_n^2 I_n, \end{cases} \tag{1.3}$$

where $c = \frac{a}{K}$.

The paper is organized as follows. It is verified that there are phenomena of the transcritical bifurcation, flip bifurcation and Hopf bifurcation in Section 2. In Section 3, a series of numerical simulations show that there are bifurcation and chaos in the discrete epidemic model. Finally, some conclusions are given.

2. Bifurcation analysis

For the Eq. (1.3), if the parameters a, b and c are fixed, by calculating, we can get the three fixed points $E_0 = (0, 0)$, $E_1 = (\frac{a}{c}, 0)$ and $E_2 = \left(\frac{1}{\sqrt{b}}, \frac{a\sqrt{b}-c}{b}\right)$. It is obvious that the fixed point E_0 is a saddle. In the following sections, we will focus on E_1, E_2.

2.1. Fixed point $E_1 = (\frac{a}{c}, 0)$

The following is the Jacobian matrix at E_1:

$$J_{E_1} = \begin{pmatrix} 1-a & -\frac{ba^2}{c^2} \\ 0 & \frac{ba^2}{c^2} \end{pmatrix},$$

where a is a bifurcation parameter. If $ba^2 = c^2$, J_{E_1} has eigenvalues $\lambda_1 = 1 - a$, $\lambda_2 = 1$. And $a \neq 2$ implies $|\lambda_1| \neq 1$. The following theorem is the case that the fixed point E_1 is a transcritical bifurcation point.

Theorem 2.1. *If $ba^2 = c^2$, $a \neq 2$, the system (1.3) will undergoes a transcritical bifurcation at E_1. Moreover, when $b > \frac{c^2}{a^2}$, the system has three fixed points, and when $b \leqslant \frac{c^2}{a^2}$, the system has two fixed points.*

Proof. Let $\xi_n = S_n - \frac{a}{c}$, $\eta_n = I_n$, $\mu_n = b - \frac{c^2}{a_2}$, and parameter μ is a new and dependent variable, the system (1.3) becomes:

$$\begin{cases} \xi_{n+1} = (1-a)\xi_n - \eta_n - c\xi_n^2 - \frac{a^2}{c^2}\mu_n\eta_n - \frac{2c}{a}\xi_n\eta_n - \frac{c^2}{a^2}\xi_n^2\eta_n - \frac{2a}{c}\xi_n\mu_n\eta_n - \mu_n\xi_n^2\eta_n, \\ \eta_{n+1} = \eta_n + \frac{2c}{a}\xi_n\eta_n + \frac{a^2}{c^2}\mu_n\eta_n + \frac{c^2}{a^2}\xi_n^2\eta_n + \mu_n\xi_n^2\eta_n + \frac{2a}{c}\xi_n\mu_n\eta_n, \\ \mu_{n+1} = \mu_n. \end{cases} \tag{2.1}$$

Let

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the following transformation:

$$\begin{pmatrix} \xi_n \\ \eta_n \\ \mu_n \end{pmatrix} = T \begin{pmatrix} u_n \\ v_n \\ \delta_n \end{pmatrix},$$