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Error bounds for the perturbation of the Drazin inverse under some geometrical conditions ☆

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ABSTRACT

This paper studies the Drazin inverse for perturbed matrices. For that, given a square matrix A, we consider and characterize the class of matrices B with index s such that $\mathscr{R}(BA^D)\cap\mathscr{N}(A^D)=\{0\}, \mathscr{N}(A^DB)\cap\mathscr{R}(A^D)=\{0\}$, and $\mathscr{R}(B^s)=\mathscr{R}(BA^D)$, where $\mathscr{N}(A)$ and $\mathscr{R}(A)$ denote the null space and the range space of a matrix A, respectively, and A^D denote the Drazin inverse of A. Then, we provide explicit representations for B^D and BB^D, and upper bounds for the relative error $\|B^D-A^D\|/\|A^D\|$ and the error $\|BB^D-AA^D\|$. A numerical example illustrates that the obtained bounds are better than others given in the literature.

1. Introduction

Let $A\in\mathbb{C}^{n\times n}$ be any complex square matrix of order n. The *index of A*, denoted by $\text{ind}(A)$, is the smallest nonnegative integer r such that $\text{rank}A^r=\text{rank}A^{r+1}$. Let $\mathscr{R}(A)$ and $\mathscr{N}(A)$ denote the range space of A and the null space of A, respectively.

We recall that the *Drazin inverse of* $A\in\mathbb{C}^{n\times n}$ is the unique matrix $A^D\in\mathbb{C}^{n\times n}$ satisfying the relations

$$A^DAA^D=A^D,\quad AA^D=A^DA,\quad A^{l+1}A^D=A^l,\ \text{for all}\ l\geqslant r,$$

where $r=\text{ind}(A)$. If A is nonsingular then $\text{ind}(A)=0$ and $A^D=A^{-1}$. The case when $\text{ind}(A)=1$, i.e., $\text{rank}A=\text{rank}A^2$, the Drazin inverse is called the *group inverse of* A and is denoted by $A^\sharp$.

If $\text{ind}(A)=r$, then $\mathbb{C}^n=\mathscr{R}(A^r)\oplus\mathscr{N}(A^r)$ and we have $\mathscr{R}(A^D)=\mathscr{R}(A^r)$ and $\mathscr{N}(A^D)=\mathscr{N}(A^r)$. The *eigenprojection of A corresponding to the eigenvalue* 0, denoted by A^π, is the uniquely determined *projector* such that $\mathscr{R}(A^\pi)=\mathscr{N}(A^r)$ and $\mathscr{N}(A^\pi)=\mathscr{R}(A^r)$.

We have that $A^\pi=I-AA^D$. Hence, we easily see that if $\text{ind}(A)=1$, then $A^\pi A=AA^\pi=O$.

In [2], Campbell and Meyer established that if A_j converges to A, then A_j^D converges to A^D if and only if $\text{rank}A_j^{r_j}=\text{rank}A^r$ for all sufficiently large j, where $r_j=\text{ind}(A_j)$. The continuity of the Drazin inverse was studied in [1–3,11].

The problem of the perturbation of the Drazin inverse for matrices has been studied recently by several authors (see [4–6,8,9,12,13]).

Let $A\in\mathbb{C}^{n\times n}$ be given and let s be a positive integer. We introduce the following geometrical conditions on $B\in\mathbb{C}^{n\times n}$:

$$\mathscr{R}(BA^D)\cap\mathscr{N}(A^D)=\{0\}, \mathscr{N}(A^DB)\cap\mathscr{R}(A^D)=\{0\},\quad \text{and}\ \mathscr{R}(B^s)=\mathscr{R}(BA^D). \tag{1.1}$$

Throughout this paper we consider the classes of perturbed matrices

$$\mathscr{A}_s=\{B\in\mathbb{C}^{n\times n}:\text{ind}(B)=s\ \text{and}\ B\ \text{satisfies conditions}\ (1.1)\}.$$

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We see that $\mathscr{A}_s$ contains the set of matrices $B \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(B) = s$ such that:

$$\mathscr{R}(B^s) = \mathscr{R}(A^{\mathrm{D}}) \quad \text{and } \mathscr{N}(B^s) = \mathscr{N}(A^{\mathrm{D}}).$$

We remark that the above conditions are equivalent to the fact that $B^\pi = A^\pi$. This latter class of matrices was characterized in [4].

Later we will prove that $\mathscr{A}_s$ also contains the class of matrices $B \in \mathbb{C}^{n\times n}$ which satisfy the following conditions:

$$\|A^{\mathrm{D}}(B - A)\| < 1, B^2AA^{\mathrm{D}} = (BAA^{\mathrm{D}})^2, \quad \text{and } \mathrm{rank} B^s = \mathrm{rank} A^{\mathrm{D}}, \tag{1.2}$$

where s is the smallest positive integer such that the third of the above conditions holds. An upper bound for the relative error $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|/\|A^{\mathrm{D}}\|$, under conditions (1.2), was derived in [9]. In this paper we extend the above mentioned result.

We note that in the case that $s = 1$, the class $\mathscr{A}_1$ can be expressed in the form

$$\mathscr{A}_1 = \{B \in \mathbb{C}^{n\times n} : \mathrm{ind}(B) = 1, \quad \mathscr{R}(B) \cap \mathscr{N}(A^D) = \{0\}, \mathscr{N}(B) \cap \mathscr{R}(A^D) = \{0\}\}.$$

The perturbation problem of group inverse is a case of special interest due to its application to the study of stability of Markov chains (see [3,10]). In this context, we prove that if $\mathscr{C}$ is an ergodic Markov chain with transition matrix $T \in \mathbb{C}^{n\times n}$ and $\widetilde{\mathscr{C}}$ is a perturbed ergodic chain of $\mathbb{C}$ with $\widetilde{T} \in \mathbb{C}^{n\times n}$ its transition matrix, then $I - \widetilde{T}$ belongs to $\mathscr{A}_1$. We obtain the results given in [10, Theorems 3.1 and 4.1].

In this paper, in Section 2, we establish several characterizations of the matrices $B \in \mathscr{A}_s$. In terms of matrix rank, this class of matrices is characterized by $\mathrm{rank} B^s = \mathrm{rank} A^{\mathrm{D}} = \mathrm{rank} A^{\mathrm{D}}B^s = \mathrm{rank} A^{\mathrm{D}}BA^{\mathrm{D}} = \dim(\mathscr{R}(BA^{\mathrm{D}}) \cap \mathscr{R}(B^{s+1}))$. We also give a block matrix representation for the perturbed matrices, which will be very useful in the perturbation analysis.

In Section 3 we provide explicit formulas of B^{D} and B^π, and we derive upper bounds for $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|/\|A^{\mathrm{D}}\|$ and $\|B^\pi - A^\pi\|$ in terms of norms involving the matrix $B - A$. We compare our bounds with others given recently in [5,12,13] in a numerical example. We prove that if B verifies (1.2), then $B \in \mathscr{A}_s$. So, we recover the result given in [9].

The next lemma establishes a condition for the existence of the group inverse of a partitioned matrix and a formula for its computation (see [3, Theorem 7.7.7]).

Lemma 1.1. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be square with* $A \in \mathbb{C}^{d\times d}$ *nonsingular and denote* $\Psi = I + A^{-1}BCA^{-1}$. *If* $\mathrm{rank} M = \mathrm{rank} A$, *then we have*

$$\mathrm{ind}(M) = 1 \iff \Psi \text{ is nonsingular}.$$

In this case, the group inverse and eigenprojection at zero of M are given by

$$M^\sharp = \begin{pmatrix} I \\ CA^{-1} \end{pmatrix} (\Psi A \Psi)^{-1} (I\ A^{-1}B),$$
$$M^\pi = \begin{pmatrix} I - \Psi^{-1} & -\Psi^{-1}A^{-1}B \\ -CA^{-1}\Psi^{-1} & I - CA^{-1}\Psi^{-1}A^{-1}B \end{pmatrix}. \tag{1.3}$$

The following lemma is concerned with the rank of a product of matrices (see [14, Sec. 2.4]).

Lemma 1.2. *Let* $A, B \in \mathbb{C}^{n\times n}$. *Then*

$$\mathrm{rank} AB = \mathrm{rank} B - \dim(\mathscr{R}(B) \cap \mathscr{N}(A)). \tag{1.4}$$

In this paper, we will use, without mention it, that if $P, Q \in \mathbb{C}^{n\times n}$ *are nonsingular, then* $\mathrm{rank} PAQ = \mathrm{rank} PA = \mathrm{rank} AQ = \mathrm{rank} A$.

2. Characterizations of the class of matrices $\mathcal{A}_s$

In this section we give several characterizations of the class of matrices $\mathscr{A}_s$. First, we recall that if $\mathrm{ind}(A) = r > 0$, then there exists a nonsingular matrix P such that we can write A in the *core-nilpotent block form*

$$A = P\begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} P^{-1} \quad A_1 \in \mathbb{C}^{d\times d} \text{ nonsingular}, \quad A_2^r = O. \tag{2.1}$$

By [3, Theorem 7.2.1], relative to the form (2.1), the Drazin inverse of A and the eigenprojection of A at zero are given by

$$A^{\mathrm{D}} = P\begin{pmatrix} A_1^{-1} & O \\ O & O \end{pmatrix} P^{-1}, \quad A^\pi = I - AA^{\mathrm{D}} = P\begin{pmatrix} O & O \\ O & I \end{pmatrix} P^{-1}. \tag{2.2}$$

We note that $\mathrm{rank} A^r = \mathrm{rank} A^D = d$.

Theorem 2.1. *Let* $A, B \in \mathbb{C}^{n\times n}$. *Assume that there exists the smallest positive integer s such that* $\mathrm{rank} B^s = \mathrm{rank} A^D = d$. *Then the following statements are equivalents:*