Applied Mathematics and Computation 213 (2009) 60–72

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A mixed finite element method for fourth order eigenvalue problems

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ARTICLE INFO

Keywords:
Mixed finite element method
Eigenvalues
Eigenvectors
Lagrange multiplier
Orthotropic problem
Vibration analysis

ABSTRACT

A mixed finite element method for approximating eigenpairs of IV order elliptic eigenvalue problems with Dirichlet boundary conditions has been given. The method can be applied to the vibration analysis of anisotropic/orthotropic/isotropic/biharmonic plates. Computer implementation procedures for this mixed method are given along with the results of numerical experiments.

1. Introduction

Bauer and Reiss [1], Canuto [2] and Ishihara [4] are probably the first papers on eigenvalue approximation of the *biharmonic eigenvalue* problem in *convex polygonal domains* using mixed finite element methods. Following the mixed method analysis of Brezzi–Raviart, error estimates for eigenvalues and eigenvectors using the mixed method schemes of Hellan–Hermann–Miyoshi [5–8] and Hellan–Hermann–Johnson [5,6,8,10] are developed in [2]. Ishihara [4] follows Miyoshi's [7] approach for the mixed method analysis of eigenpairs. In [13], the authors have made an abstract analysis of eigenvalue and eigenvector approximations of *biharmonic* eigenvalue problems using mixed/hybrid finite element methods in a very general setting by a compact operator approach. In [14], a mixed method analysis for *general fourth order elliptic* eigenvalue problems defined on *convex domains* has been developed taking into account the effect of numerical integration and boundary approximation using isoparametric mapping. Results on the parallel implementation of the eigenvalue problem and that of the corresponding source problem solved using this mixed method have been developed in [15] and [17], respectively. Andreev et al. [3] deal with a new mixed finite element method and use a post processing technique for obtaining a higher order convergence for the eigenvalues of the *biharmonic* eigenvalue problem at the expense of solving an additional source problem on an augmented finite element space.

In this paper, another mixed finite element method for general *fourth order* elliptic *eigenvalue* problems defined on *convex polygonal* domains has been developed. This method which is applicable to the vibration analysis of anisotropic/orthotropic/isotropic/biharmonic plates has been derived from the *Hellan–Hermann–Johnson* scheme for the *biharmonic eigenvalue problems* developed in [2]. In fact, the error estimates for eigenpairs of the general fourth order eigenvalue problem using the mixed method developed in this paper can be established following the ideas which are systematically and elaborately developed in [14,18]. But this mixed method (and also the mixed method developed for biharmonic eigenvalue problem [2]) is **not** directly suitable for computer implementation purposes. This is because, in this method, the definition of the admissible space of tensor field in the continuous mixed variational (resp. mixed finite element) formulation demands the continuity of $M_n(\Phi) = \sum_{i,j=1}^{2} \phi_{ij} n_i n_j$ (resp. $M_n(\Phi_h) = \sum_{i,j=1}^{2} \phi_{hij} n_i n_j$) along the interelement boundaries of the triangulation τ_h of $\bar{\Omega}$ (see (2.5) and (2.11)). Since it is difficult to incorporate this condition in the implementation procedure, a *modified mixed finite element method* is developed in which the interelement continuity constraint (see (2.11)) in the discrete problem is relaxed by introducing a suitable Lagrange multiplier. Results of parallelization of the corresponding source problem has

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2009.02.050

been presented in [16]. It is also shown that this modified mixed finite element scheme is equivalent in some sense (to be introduced in the sequel) to the original mixed finite element method for which error estimates for eigenpairs under suitable regularity assumptions on eigenvectors can be obtained exactly using the analysis in [2] for biharmonic problems and [14] for anisotropic/orthotropic/isotropic plate vibration problems.

From the implementation point of view, the modified mixed finite element method for finding eigenpairs is very simple and economical *in comparison* with the mixed method scheme developed in [14,15,18]. In the scheme developed in [14,15,18], the construction of the global stiffness matrix assembling the element matrices was very expensive as the admissible space of the tensor fields were globally continuous functions and hence a local assembly of the components of tensor fields was not possible. But in this case, the basis functions for the tensor field need not be globally continuous functions and hence the assembly process is easier. The basis functions are *not* associated with the nodes of the triangulation and can be assumed independently in each triangle.

The outline of the paper is as follows. In Section 2, the continuous mixed variational, the mixed finite element and the modified mixed finite element eigenvalue problems are defined. The equivalence of the modified mixed finite element method with the mixed finite element method is established. Section 3 gives the computer implementation procedure. In Section 4, the results of some numerical experiments which establish the efficacy of the modified mixed method are presented.

2. The mixed variational and finite element eigenvalue problems

Consider the following eigenvalue problem:
Find $\lambda \in \mathbb{R}^+$, for which $\exists$non-null u such that

$$(P_E): \begin{bmatrix} \Lambda u = \lambda u & \text{in } \Omega, \\ u|_\Gamma = 0, & \frac{\partial u}{\partial n}\big|_\Gamma = 0, \end{bmatrix} \tag{2.1}$$

where the fourth order operator Λ is defined by

$$\begin{aligned}(\Lambda u)(x) &= \frac{\partial^2}{\partial x_k \partial x_l}\left(a_{ijkl}\frac{\partial^2 u}{\partial x_i \partial x_j}\right)(x) \quad \forall x \in \Omega \\ &= (a_{ijkl}u_{,ij})_{,kl}(x) \quad \forall x \in \Omega,\end{aligned} \tag{2.2}$$

Ω being a convex domain in $\mathbb{R}^2$ with boundary Γ. (In (2.2) and in the sequel, Einstein's summation convention with respect to twice repeated indices $i,j,k,l = 1,2$ has been followed unless otherwise stated.)

The coefficients $a_{ijkl},\ 1 \leqslant i,j,k,l \leqslant 2$, satisfy the following assumptions: $\forall i,j,k,l = 1,2$ and $\forall x = (x_1,x_2) \in \overline{\Omega}$,

$$a_{ijkl}(x) \in \mathscr{C}^0(\overline{\Omega}), \quad a_{ijkl}(x) = a_{klij}(x) = a_{klji}(x) = a_{lkji}(x), \tag{2.3}$$

$$\exists \alpha > 0 \ni \forall \underline{\xi} = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) \in \mathbb{R}^4 \quad \text{with } \xi_{12} = \xi_{21}, \quad a_{ijkl}(x)\xi_{ij}\xi_{kl} \geqslant \alpha \|\underline{\xi}\|^2_{\mathbb{R}^4}. \tag{2.4}$$

Note that, for suitable choices of the coefficients a_{ijkl}, (2.1) corresponds to the vibration problem corresponding to anisotropic/orthotropic/isotropic/biharmonic plates which are very *thin*, i.e. the thickness is much smaller than the other two dimensions.

For formulating the continuous mixed variational eigenvalue problem, we need the following:

- Let τ_h be an admissible regular, quasi-uniform triangulation [12,19] of the domain $\overline{\Omega}$ into closed triangles T.
- $\mathbf{H} = \{\Phi : \Phi = (\phi_{ij})_{1 \leqslant i,j \leqslant 2}, \phi_{ij} \in L^2(\Omega), \phi_{12} = \phi_{21}\}$.
- For $\Phi \in \mathbf{H}$ with $\phi_{ij}|_T \in H^1(T)\ \forall T \in \tau_h,\ M_n(\Phi) = \phi_{ij}n_in_j$ is said to be continuous at the interelement boundaries of the triangulation τ_h, iff for any pair T_1, T_2 of adjacent triangles of τ_h with a common side $T_1 \cap T_2$,

  $$M_{\mathbf{n_1}}(\Phi|_{T_1}) = M_{\mathbf{n_2}}(\Phi|_{T_2}) \quad \text{on } T_1 \cap T_2 = \partial T_1 \cap \partial T_2,$$

  where $\mathbf{n_i}$ is the unit exterior normal to the boundary ∂T_i of $T_i,\ i = 1,2$.
- $$\begin{aligned}\mathbf{V} = \{&\Phi : \Phi = (\phi_{ij})_{1 \leqslant i,j \leqslant 2}, \Phi \in \mathbf{H}, \phi_{ij}|_T \in H^1(T)\ \forall T \in \tau_h, \\ &M_n(\Phi) \text{ is continuous across interelement boundaries of } \tau_h\ \}\end{aligned} \tag{2.5}$$
- $$W = W_0^{1,p}(\Omega) = \{\chi \in W^{1,p}(\Omega), \chi|_\Gamma = 0\}, \quad p > 2. \tag{2.6}$$
- The continuous, symmetric bilinear form $A(\cdot,\cdot) : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$ is defined by:

  $$A(\Psi, \Phi) = \int_\Omega A_{ijkl}\psi_{ij}\phi_{kl}\, d\Omega, \tag{2.7}$$

  where $A_{ijkl} = A_{ijkl}(x)$ are new coefficients defined in terms of coefficients $a_{ijkl}(x)$ (see [20] for the details of the algorithm).
- $$\exists \alpha > 0 \text{ such that } A(\Phi, \Phi) \geqslant \alpha \sum_{i,j=1}^{2} \|\phi_{ij}\|^2_{0,\Omega} \quad \forall \Phi \in \mathbf{H}. \tag{2.8}$$