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Efficient L-stable method for parabolic problems with application to pricing American options under stochastic volatility

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ABSTRACT

Efficient L-stable numerical method for semilinear parabolic problems with nonsmooth initial data is proposed and implemented to solve Heston's stochastic volatility model based PDE for pricing American options under stochastic volatility. The proposed new method is also used to solve two asset American options pricing problem. Cox and Matthews [S.M. Cox, P.C. Matthews, Exponential time differencing for stiff systems, Journal of Computational Physics 176 (2002) 430–455] developed a class of exponential time differencing Runge–Kutta schemes (ETDRK) for nonlinear parabolic problems. Kassam and Trefethen [A.K. Kassam, L.N. Trefethen, Fourth-order time stepping for stiff PDEs, SIAM Journal on Scientific Computing 26 (4) (2005) 1214–1233] showed that while computing certain functions involved in the Cox–Matthews schemes, severe cancelation errors can occur which affect the accuracy and stability of the schemes. Kassam and Trefethen proposed complex contour integration technique to implement these schemes in a way that avoids these cancelation errors. But this approach creates new difficulties in choosing and evaluating the contour integrals for larger problems. We modify the ETDRK schemes using positivity preserving Padé approximations of the matrix exponential functions and construct computationally efficient parallel version using splitting technique. As a result of this approach it is required only to solve several backward Euler linear problems in serial or parallel.

1. Introduction

Problems having irregular initial data or mismatched initial and boundary conditions occur in various applications, including mechanical engineering, computational chemistry, and financial engineering. Nonsmooth payoffs cause discontinuities in the solution (or its derivatives) and standard A-stable methods (e.g., Crank–Nicolson) are prone to produce large and spurious oscillations in the numerical solutions which would mislead to estimating options accurately if one does not treat the problem carefully.

Cox and Matthews [2] developed a class of exponential time differencing schemes (ETD) for nonlinear stiff systems of ODEs and extended the results to solve nonlinear parabolic problems. This approach reduces the spatially discretized PDE using Duhamel's principle on one time step to an integral equation followed by approximation of the integral involving nonlinear function. The nonlinear function is approximated by a polynomial. In the analysis, Cox and Matthews treated scalar examples (ODEs) and systems of two PDEs with special form. Kassam and Trefethen [9] addressed the limited generality of the Cox–Matthews schemes and showed that these schemes can suffer from severe cancelation errors when computing certain functions involved in the schemes. A new strategy based on contour integral was introduced to improve

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the general applicability of these schemes. Use of complex contour integration to implement these schemes avoids inverting matrix polynomials, but this approach creates new difficulties in choosing and evaluating the contour integrals for larger problems.

Cox–Matthews schemes as well as Kassam–Trefethen modifications require calculating matrix exponentials. Even if the original matrix is sparse, the matrix exponential will not itself be sparse, which can be a significant amount of work and affect the computational efficiency of the scheme. For one-dimensional problems the calculation is not very expensive, however, the cost goes up as the dimension increases. Also, neither Cox–Matthews development nor Kassam–Trefethen modification addresses problems with nonsmooth data.

In the Kassam–Trefethen approach, contour integrals are evaluated by means of some numerical technique and must contain spectrum of the discretization matrix A. Generally the eigenvalues of A lie in or near the left half of the complex plane and they may cover a wide range which grows with spatial mesh size N. The spectrum of the discretization matrix A is not easily known and it is typically unbounded as the spatial step goes to zero. This is a primary limitation of the Kassam–Trefethen modification because the contour varies from problem to problem, with dependence on the spatial mesh. This limitation makes the technique problem dependent. For example, eigenvalues for diffusive problems are close to the negative real axis and for dispersive problems they are close to the imaginary axis.

Another recent development in this area is implementation of the second order exponential time differencing scheme to the magnetohydrodynamic equations in a spherical shell. A variety of different methods including direct computation, contour integration, spectral expansions and recurrence relations are discussed and implemented, see [16].

Main contribution in this article is to develop an alternate solution to these computational difficulties. A second order ETDRK scheme requires inverting a second degree matrix polynomial where as third and fourth order schemes require inverting cubic matrix polynomials, which can cause serious numerical instability and computational difficulties because of the ill-conditioning (see [5, Section 6.2]). We modify Cox–Matthews schemes to the general nondiagonal problems using Padé approximations of the matrix exponential functions and use splitting technique to construct parallel versions of the schemes. This approach transforms the matrix polynomial inversion problem into a sum of well-conditioned linear problems that can be solved in parallel. Our formulation of the modified schemes is generally more accurate for problems with irregular data and computationally more efficient as compared to the aforementioned ETDRK schemes.

A fourth order L-stable method is constructed using positivity preserving sub-diagonal (0,4)-Padé approximation. To show the advantage of L-stable method, we have constructed an A-stable method using diagonal (2,2)-Padé approximation. An algorithm based on the modified schemes is developed and implemented to solve two important problems from financial mathematics. Heston's stochastic volatility model with a small nonlinear penalty term is used for pricing American put options under stochastic volatility. Penalty method was first introduced by Zvan et al. [24] for American options under stochastic volatility. Forsyth and Vetzal [3] proposed an implicit finite difference scheme for valuing American options using the penalty method. Nielsen et al. [17] presented a refinement of Zvan's work and illustrated the performance of various numerical schemes using explicit, semi-implicit, and fully implicit methods. Khaliq et al. [12] used linearly implicit predictor–corrector schemes for pricing American options.

Organization of this paper is as follows. In Section 2 we consider an abstract PDE and write its solution using Duhamel's principle. Basic time stepping schemes are given in Section 3 and Padé approximations as well as modified schemes are mentioned in Section 4. Parallel implementation of these schemes with a parallel algorithm is given in Section 5. Models for pricing American options and penalty method approach is described in Section 6. Section 7 contains an efficient spatial discretization approach. Numerical results are discussed in Section 8. Finally we provide some concluding remarks in Section 9.

2. The abstract PDE

We consider the following semilinear initial-boundary value problem:

$$\begin{aligned} &u_t + Au = F(u,t) \;\text{ in } \Omega, \quad t \in (0,\bar{t}] = J, \\ &u = v \;\text{ on } \partial\Omega, \quad t \in J, \qquad u(\cdot,0) = u_0 \;\text{ in } \Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded domain in $\mathbf{R}^d$ with Lipschitz boundary, F is a smooth nonlinear function defined on $\mathbf{R}^d$ and A denotes a uniformly elliptic operator,

$$Au := -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}\left(a_{j,k}\frac{\partial u}{\partial x_k}\right) + \sum_{j=1}^{d} b_j \frac{\partial u}{\partial x_j} + b_0 u. \tag{2.2}$$

The coefficients $a_{j,k}$ and b_j are C^∞ (or sufficiently smooth) functions on $\overline{\Omega}, a_{j,k} = a_{k,j}, b_0 \geqslant 0$, and for some $c_0 > 0$

$$\sum_{j,k=1}^{d} a_{j,k}\xi_j\xi_k \geqslant c_0|\xi|^2 \;\text{ on } \overline{\Omega}, \quad \text{for all } \xi \in \mathbf{R}^d. \tag{2.3}$$

The initial-value problem (2.1) is considered in a Hilbert space $\mathscr{H}$, with A being a linear, self-adjoint, positive definite closed operator with a compact inverse T, defined on a dense domain $D(A) \subset \mathscr{H}$, see [21] for more details. The operator A could represent any of $\{A_h\}_{0<h\leqslant h_0}$, obtained from a spatial discretization and $\mathscr{H}$ could be an appropriate finite-dimensional subspace of $L_2(\Omega)$, cf. [19,21].