Applied Mathematics and Computation 206 (2008) 687–694

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A note on the numerical approximate solutions for generalized Sylvester matrix equations with applications

A. Bouhamidi *, K. Jbilou

L.M.P.A, Université du Littoral, 50 rue F. Buisson BP699, F-62228 Calais Cedex, France

ARTICLE INFO

Keywords:
Linear matrix equation
Global GMRES
Matrix Krylov subspace
Preconditioning

ABSTRACT

In the present paper, we propose a Krylov subspace method for solving large and sparse generalized Sylvester matrix equations. The proposed method is an iterative projection method onto matrix Krylov subspaces. As a particular case, we show how to adapt the ILU and the SSOR preconditioners for solving large Sylvester matrix equations. Numerical examples and applications to some PDE's will be given.

1. Introduction

The aim of this paper is to present numerical Krylov subspace methods for solving the linear matrix equation

$$\sum_{i=1}^{q} A_i X B_i = C, \tag{1}$$

where $A_i \in \mathbb{R}^{n\times n}$; $B_i \in \mathbb{R}^{p\times p}$, $i = 1,\ldots,q$; C and $X \in \mathbb{R}^{n\times p}$.

Such problems arise in the solution of large eigenvalue problems [6] and in the boundary value problem. They play also an important role in linear control and filtering theory for continuous or discrete-time large-scale dynamical systems, image restoration and other problems; see [2–5,8,11–13,16–19] and the references therein. The matrix equation (1) contains the well-known Lyapunov, Sylvester and Stein matrix equations.

The linear matrix equation can be written as the following $np \times np$ linear system:

$$\left[\sum_{i=1}^{q} (B_i^{\mathrm{T}} \otimes A_i)\right] \mathrm{vec}(X) = \mathrm{vec}(C), \tag{2}$$

where $\mathrm{vec}(X)$ is the vector of $\mathbb{R}^{np}$ obtained by stacking the columns of the $n \times p$ matrix X and $\otimes$ denotes the Kronecker product; $(F \otimes G = [f_{ij}G])$. Krylov subspace methods such as the GMRES algorithm [15] could be used to solve the linear system (2). However, for large problems this approach cannot be applied directly.

In the present paper, we present a global approach for solving the matrix equation (1). Our method uses the global generalized minimal residual (GlGMRES) method [10] which was originally introduced for solving linear systems with multiple right-hand sides.

In Section 2, we recall the global generalized minimal residual (GlGMRES) method, show how to apply the GlGMRES method for solving the matrix equation (1) and give some theoretical results. In Section 3, we a left-right preconditioner

* Corresponding author.
E-mail addresses: bouhamidi@lmpa.univ-littoral.fr (A. Bouhamidi), jbilou@lmpa.univ-littoral.fr (K. Jbilou).

0096-3003/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2008.09.022

for large Sylvester matrix equations. Section 4 is devoted to the symmetric successive overrelaxation (SSOR) preconditioning in association with the GlGMRES algorithm for Sylvester matrix equations. In the last section we give some numerical experiments.

In this paper, we use the following notations. For X and Y two matrices in $\mathbb{R}^{n\times p}$, we define the following inner product $\langle X, Y\rangle_F = tr(X^{\mathrm{T}}Y)$ where $tr(.)$ denotes the trace and X^{T} the transpose of the matrix X. The associated norm is the well known Frobenius norm denoted by $\|\cdot\|_F$. For $V \in \mathbb{R}^{n\times p}$, the matrix Krylov subspace $\mathscr{K}_k(A, V)$ is the subspace generated by the vectors (matrices) $V, AV, \ldots, A^{k-1}V$. Unless specified, the Frobenius norm will be used for matrices and vectors. A system of matrices of $\mathbb{R}^{n\times p}$ is said to be F-orthogonal if it is orthogonal with respect to the scalar product $\langle ., .\rangle_F$.

2. The global-GMRES method for linear matrix equations

In this section, we present a numerical Krylov subspace method for solving the linear matrix equation (1). Eq. (1) has a unique solution if and only if the matrix $\sum_{i=1}^{q} B_i^{\mathrm{T}} \otimes A_i$ is nonsingular. Throughout this paper, we assume that this condition is verified.

Let $\mathscr{M}$ be the operator defined as follows:

$$\mathscr{M} : \mathbb{R}^{n\times p} \to \mathbb{R}^{n\times p},$$

$$X \to \sum_{i=1}^{q} A_i X B_i.$$

The transpose of the operator $\mathscr{M}$ with respect to the inner product $\langle ., .\rangle_F$ is defined from $\mathbb{R}^{n\times p}$ onto $\mathbb{R}^{n\times p}$ by $\mathscr{M}^{\mathrm{T}}(X) = \sum_{i=1}^{q} A_i^{\mathrm{T}} X B_i^{\mathrm{T}}$. Next, we show how to solve iteratively the problem (1) using Krylov subspace methods.

Let V be any $n \times p$ matrix and consider the matrix Krylov subspace associated to the pair $(\mathscr{M}, V)$ and an integer k defined by

$$\mathscr{K}_k(\mathscr{M}, V) = \mathrm{span}\{V, \mathscr{M}(V), \ldots, \mathscr{M}^{k-1}(V)\}.$$

We note that $\mathscr{M}^i(V)$ is defined recursively as $\mathscr{M}^i(V) = \mathscr{M}(\mathscr{M}^{i-1}(V))$. Remark that the matrix Krylov subspace $\mathscr{K}_k(\mathscr{M}, V)$ is a subspace of $\mathbb{R}^{n\times p}$.

The modified global Arnoldi algorithm [10] constructs an F-orthonormal basis $V_1, V_2, \ldots, V_k$ of the matrix Krylov subspace $\mathscr{K}_k(\mathscr{M}, V)$, i.e.

$$\langle V_i, V_j\rangle_F = \delta_{ij}, \quad \text{for } i, j = 1, \ldots, k,$$

where δ_{ij} denotes the classical Kronecker symbol. The algorithm is described as follows:

Algorithm 1 (*Modified Global Arnoldi algorithm*).

1. Set $V_1 = V/\|V\|_F$.
2. For $j = 1, \ldots, k$. do
  $\widetilde{V} = \mathscr{M}(V_j)$,
  for $i = 1, \ldots, j$. do
    $h_{ij} = \langle V_i, \widetilde{V}\rangle_F$,
    $\widetilde{V} = \widetilde{V} - h_{ij}V_i$,
  endfor
  $h_{j+1,j} = \|\widetilde{V}\|_F$,
  $V_{j+1} = \widetilde{V}/h_{j+1,j}$.
EndFor.

Let $\mathscr{V}_k$ be the $n \times kp$ matrix: $\mathscr{V}_k = [V_1, V_2, \ldots, V_k]$. $\widetilde{H}_k$ denotes the $(k+1) \times k$ upper Hessenberg matrix whose nonzero entries h_{ij} are defined by Algorithm 1 and H_k is the $k \times k$ matrix obtained from $\widetilde{H}_k$ by deleting its last row. Note that the block matrix $\mathscr{V}_k$ is F-orthonormal which means that the matrices $V_1, \ldots, V_k$ are orthonormal with respect to the scalar product $\langle \cdot, \cdot\rangle_F$.

It is not difficult to show the following proposition:

Proposition 1. *We have the following relations:*

1. $[\mathscr{M}(V_1), \ldots, \mathscr{M}(V_k)] = \mathscr{V}_k(H_k \otimes I_p) + E_{k+1}$, *where* $E_{k+1} = h_{k+1,k}[0_{n\times p}, \ldots, 0_{n\times p}, V_{k+1}]$.
2. $[\mathscr{M}(V_1), \ldots, \mathscr{M}(V_k)] = \mathscr{V}_{k+1}(\widetilde{H}_k \otimes I_p)$.
3. *For any* $(k+1) \times s$ *matrix* G, *we have* $\|\mathscr{V}_{k+1}(G \otimes I_p)\|_F = \|G\|_F$.

Starting from an initial guess $X_0 \in \mathbb{R}^{n\times p}$ and the corresponding residual $R_0 = C - \mathscr{M}(X_0)$, the global GMRES method defines, at step k, the approximate solution X_k as follows: