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The Drazin inverse of bounded operators with commutativity up to a factor ☆

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ABSTRACT

We explore the Drazin inverse of bounded operators with commutativity up to a factor, $PQ = \lambda QP$, in a Banach space. Conditions on Drazin invertibility are formulated and shown to depend on spectral properties of the operators involved. We also present a result concerning the more general problem of commutativity up to a related operator factor, $PQ = PQP$. Under the condition of commutativity up to a factor $PQ = \lambda QP$ (resp. $PQ = PQP$), we give explicit representations of the Drazin inverse $(P-Q)^{\mathrm{D}}$ (resp. $(P+Q)^{\mathrm{D}}$) in term of P, P^{D}, Q and Q^{D}.

1. Introduction

Commutation relations between operators in a Banach space X play an important role in the representations of the Drazin inverse. Some properties of the Drazin inverse according to such relations have been extensively studied in the mathematical literature (see, e.g. [16,18]). An interesting related aspect is to consider the Drazin inverse $(P-Q)^{\mathrm{D}}$ on the condition of the commutativity up to a factor, i.e.,

$$PQ = \lambda QP, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

We also consider the Drazin inverse $(P+Q)^{\mathrm{D}}$ concerning the more general problem of commutativity up to a related operator factor, i.e.,

$$PQ = PQP.$$

It is well known that, if $PQ = QP = 0$, Drazin in his celebrated paper [18] had proved that

$$(P-Q)^{\mathrm{D}} = P^{\mathrm{D}} - Q^{\mathrm{D}};$$

if $PQ = 0$, Hartwig et al. for matrices (see [21]), Djordjević and Wei for operators (see [16]) had proved that

$$(P+Q)^{\mathrm{D}} = Q^{\mathrm{D}}\left[\sum_{n=0}^{s-1}(Q^{\mathrm{D}})^n P^n\right](I - PP^{\mathrm{D}}) + (I - QQ^{\mathrm{D}})\left[\sum_{n=0}^{t-1} Q^n (P^{\mathrm{D}})^n\right]P^{\mathrm{D}}.$$

Using the technique of block operator matrices, we will investigate explicit representations of the Drazin inverse $(P \pm Q)^{\mathrm{D}}$ in term of P, P^{D}, Q and Q^{D} under the condition of $PQ = \lambda QP$ or $PQ = PQP$. The connection of our formulae to those exist results is also explored.

We adopt the following definitions and notations.

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Let $\mathscr{B}(X)$ be the Banach algebra of all bounded linear operators on X. For an operator $T \in \mathscr{B}(X)$, the symbols $\mathscr{N}(T), \mathscr{R}(T)$ and $\sigma(T)$ will denote the null space, the range space and the spectrum of T, respectively. For $T \in \mathscr{B}(X)$, if there exists an operator $T^{\mathrm{D}} \in \mathscr{B}(X)$ satisfying the following three operator equations [18]:

$$TT^{\mathrm{D}} = T^{\mathrm{D}}T, \quad T^{\mathrm{D}}TT^{\mathrm{D}} = T^{\mathrm{D}}, \quad T^{k+1}T^{\mathrm{D}} = T^k, \tag{1}$$

where $k = \mathrm{ind}(T)$, the index of T, is the smallest nonnegative integer for which $\mathscr{R}(T^{k+1}) = \mathscr{R}(T^k)$ and $\mathscr{N}(T^{k+1}) = \mathscr{N}(T^k)$, then T^{D} is called a Drazin inverse of T [1,3,4,6–15,17,19,20,23–30]. In particular, when $1 = \mathrm{ind}(T)$, the operator T^{D} is called the group inverse of T, and is denoted by $T^{\#}$. The conditions (1) are equivalent to

$$TT^{\mathrm{D}} = T^{\mathrm{D}}T, \quad T^{\mathrm{D}}TT^{\mathrm{D}} = T^{\mathrm{D}}, \quad T - T^2T^{\mathrm{D}} \text{ is nilpotent}. \tag{2}$$

If T is Drazin invertible, then the spectral idempotent P of T corresponding to $\{0\}$ is given by $P = I - TT^{\mathrm{D}}$. The operator matrix form of T with respect to the space decomposition $X = \mathscr{N}(P) \oplus \mathscr{R}(P)$ is given by $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent. Throughout the paper we let I represent the identity operator on its domain.

2. Pairs commuting up to a scalar factor

We first obtain the following useful observations.

Lemma 2.1. *Let $PQ = \lambda QP$ and $PQ \neq 0$. Then*

(1) $PQ^i = \lambda^i Q^i P$ *and* $P^i Q = \lambda^i QP^i$.
(2) *If P or Q is invertible, then*

$$(P^{-1}Q)^i = \lambda^{\sum_{k=0}^{i-1} k} P^{-i}Q^i, (PQ^{-1})^i = \lambda^{\sum_{k=0}^{i-1} k} P^i Q^{-i}.$$

Lemma 2.2 ([2]). *Let $PQ = \lambda QP$ and $PQ \neq 0$. Then*

(1) $\sigma(P+Q) \cup \sigma(\lambda Q) = \sigma(Q) \cup \sigma(P + \lambda Q), \sigma(PQ) = \sigma(QP) = \lambda\sigma(PQ)$. *In particular, if $0 \notin \sigma(PQ)$, both P and Q have bounded inverses.*
(2) *If one of P or Q is nilpotent, then $\sigma(PQ) = \sigma(QP) = \{0\}$;*
(3) *If both P and Q are nilpotent, then $\sigma(P+Q) = \{0\}$.*

Proof

(1) For any $\mu \in \mathbb{C}$, if $PQ = \lambda QP$, then

$$(\mu I - \lambda Q)(\mu I - P - Q) = (\mu I - P - \lambda Q)(\mu I - Q).$$

This implies that

$$\sigma(P+Q) \cup \sigma(\lambda Q) = \sigma(Q) \cup \sigma(P + \lambda Q).$$

Since $PQ \neq 0$, we have $\lambda \neq 0$. Suppose $0 \in \sigma(PQ)$ and $0 \notin \sigma(QP)$. Then QP is invertible, and $PQ(QP)^{-1} = (QP)^{-1}PQ = \lambda I$. From $0 \in \sigma(PQ)$ we obtain that $\lambda = 0$. This is a contradiction. Similarly, $0 \notin \sigma(PQ)$ and $0 \in \sigma(QP)$ cannot occur. Hence, either $0 \in \sigma(PQ) \cap \sigma(QP)$ or $0 \notin \sigma(PQ) \cup \sigma(QP)$. Then $\sigma(PQ) = \sigma(QP)$. If $0 \notin \sigma(PQ) = \sigma(QP)$, then PQ and QP have bounded inverses since their spectra are bounded away from 0. Hence both P and Q are surjective and injective and thus they have bounded inverses.

(2) Let P be nilpotent. Then there exists a positive integer k such that $P^k = 0$. From $PQ = \lambda QP$ we have $(QP)^k = \lambda^{\sum_{i=0}^{k-1} i} Q^k P^k = 0$. Then $\sigma(PQ) = \sigma(QP) = \{0\}$. Similarly, we can prove that $\sigma(PQ) = \sigma(QP) = \{0\}$ if Q is nilpotent.

(3) If P and Q are nilpotent, then $(P+Q)^2 = P^2 + (\lambda+1)QP + Q^2$. A direct computation can show that

$$(P+Q)^k = P^k + f_1(\lambda)QP^{k-1} + f_2(\lambda)Q^2P^{k-2} + \cdots + f_{k-1}(\lambda)Q^{k-1}P + Q^k,$$

where $f_i(\lambda) (i = 1, 2, \ldots, k-1)$ are the polynomial of λ. So, if $k \geqslant \mathrm{ind}(P) + \mathrm{ind}(Q) - 1$, then $(P+Q)^k = 0$ and $\sigma(P+Q) = \{0\}$. □

Theorem 2.3. *Let $P, Q \in \mathscr{B}(X)$ be Drazin invertible. If $PQ = \lambda QP$ and $PQ \neq 0$, then*

(1) $(PQ)^{\mathrm{D}} = Q^{\mathrm{D}}P^{\mathrm{D}} = \frac{1}{\lambda}P^{\mathrm{D}}Q^{\mathrm{D}}$.
(2) $PQ^{\mathrm{D}} = \frac{1}{\lambda}Q^{\mathrm{D}}P$.
(3) $P^{\mathrm{D}}Q = \frac{1}{\lambda}QP^{\mathrm{D}}$.