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Fractional calculus with an integral operator containing a generalized Mittag–Leffler function in the kernel

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ABSTRACT

In this paper, we introduce and investigate a fractional calculus with an integral operator which contains the following family of generalized Mittag–Leffler functions:

$$E_{\alpha,\beta}^{\gamma,\kappa}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}\quad (z,\beta,\gamma\in\mathbb{C};\ \Re(\alpha)>\max\{0,\Re(\kappa)-1\};\ \Re(\kappa)>0)$$

in its kernel, $(\lambda)_\nu$ being the familiar Pochhammer symbol. A number of corollaries and consequences of the main results presented here are also considered.

1. Introduction, definitions and preliminaries

As mentioned in the survey article by Gorenflo and Mainardi [4], even the *classical* Mittag–Leffler functions have remained, for a long time, almost totally ignored in the common handbooks on special functions and tables of integral transforms, although a reasonably adequate description of many of their interesting and potentially useful properties appeared already in the *third* volume of the celebrated Bateman Manuscript Project (see Erdélyi et al. [2, Chapter 18]) in a chapter devoted to *Miscellaneous Functions*). The Mittag–Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are defined by the following series:

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}\quad (z\in\mathbb{C};\ \Re(\alpha)>0) \tag{1.1}$$

and

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}\quad (z,\beta\in\mathbb{C};\ \Re(\alpha)>0), \tag{1.2}$$

respectively. These functions are natural extensions of the exponential, hyperbolic and trigonometric functions, since

$$E_1(z)=e^z,\quad E_2(z^2)=\cosh z\quad\text{and}\quad E_2(-z^2)=\cos z.$$

For a detailed account of the various properties, generalizations and applications of the Mittag–Leffler functions, the reader may refer to the recent works by (for example) Džrbašjan [1], Kilbas and Saigo [11], Gorenflo and Mainardi [4], Gorenflo et al.

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([5–7]), Kilbas et al. [13, Chapter 1], and Saigo and Kilbas [17]. The Mittag–Leffler function (1.1) and some of its various generalizations have only recently been calculated numerically in the whole complex plane (see, for example, [10,19]). By means of the series representation, a generalization of the Mittag–Leffler function $E_{\alpha,\beta}(z)$ of (1.2) was introduced by Prabhakar [16] as follows:

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}\quad (z,\beta,\gamma\in\mathbb{C};\Re(\alpha)>0), \tag{1.3}$$

where (*and throughout this investigation*) $(\lambda)_\nu$ denotes the familiar Pochhammer symbol or the *shifted* factorial, since

$$(1)_n=n!\quad (n\in\mathbb{N}_0),$$

defined (for $\lambda,\nu\in\mathbb{C}$ and in terms of the familiar Gamma function) by

$$(\lambda)_\nu:=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}=\begin{cases}1 & (\nu=0;\lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\lambda\in\mathbb{C})\end{cases} \tag{1.4}$$
$$(\mathbb{N}_0:=\mathbb{N}\cup\{0\};\mathbb{N}:=\{1,2,3,\ldots\}).$$

Clearly, we have the following special cases:

$$E^1_{\alpha,\beta}(z)=E_{\alpha,\beta}(z)\quad\text{and}\quad E^1_{\alpha,1}(z)=E_\alpha(z). \tag{1.5}$$

Indeed, as already observed earlier by Srivastava and Saxena [28, p. 201, Eq. (1.6)], the generalized Mittag–Leffler function $E^{\rho}_{\alpha,\beta}(z)$ itself is actually a very specialized case of a rather extensively investigated function ${}_p\Psi_q$ as indicated below (see also [13, p. 45, Eq. (1.9.1)]):

$$E^{\gamma}_{\alpha,\beta}(z)=\frac{1}{\Gamma(\gamma)}\,{}_1\Psi_1\left[\begin{matrix}(\gamma,1);\\ \\ (\beta,\alpha);\end{matrix}\ z\right]. \tag{1.6}$$

Here, *and in what follows*, ${}_p\Psi_q$ denotes the Wright (or, *more appropriately*, the Fox–Wright) generalization of the hypergeometric ${}_pF_q$ function, which is defined by (see, for example, [25, p. 21])

$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\\ \\ (b_1,B_1),\cdots,(b_q,B_q);\end{matrix}\ z\right]:=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+A_1k)\cdots\Gamma(a_p+A_pk)}{\Gamma(b_1+B_1k)\cdots\Gamma(b_q+B_qk)}\frac{z^k}{k!} \tag{1.7}$$
$$\left(\Re(A_j)>0\ (j=1,\ldots,p);\Re(B_j)>0\ (j=1,\ldots,q);1+\Re\left(\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\right)\geqq 0\right),$$

in which we have assumed, in general, that

$$a_j,A_j\in\mathbb{C}\ \ (j=1,\cdots,p)\quad\text{and}\quad b_j,B_j\in\mathbb{C}\ \ (j=1,\ldots,q)$$

and that the equality in the convergence condition holds true only for suitably bounded values of $|z|$. In fact, we have (see, for details, [23, p. 19]; see also [13, p. 58 *et seq.*])

$${}_p\Psi_q\left[\begin{matrix}(a_1,1),\ldots,(a_p,1);\\ \\ (b_1,1),\ldots,(b_q,1);\end{matrix}\ z\right]=\frac{\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_1)\cdots\Gamma(b_q)}\ {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ \\ b_1,\ldots,b_q;\end{matrix}\ z\right] \tag{1.8}$$

and

$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\\ \\ (b_1,B_1),\ldots,(b_q,B_q);\end{matrix}\ z\right]=H^{1,p}_{p,q+1}\left[-z\left|\begin{matrix}(1-a_1,A_1),\ldots,(1-a_p,A_p)\\ (0,1),(1-b_1,B_1),\ldots,(1-b_q,B_q)\end{matrix}\right.\right] \tag{1.9}$$

in terms of the familiar F and H functions, respectively.

Remark 1. Throughout this paper, it is tacitly assumed that the complex parameter β in the definitions (1.2) and (1.3) as well as the complex parameters $b_1,\cdots,b_q$ in the definition (1.7) are so constrained that no zeros appear in the denominators on the right-hand sides of (1.2), (1.3) and (1.7).

Remark 2. By recalling the following definition of the *classical* Riemann–Liouville fractional derivative operator $\mathcal{D}^{\mu}_z$ (see, for example, the works by Kilbas et al. [13] and Samko et al. [18]):