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Integral-type operators acting between weighted-type spaces on the unit ball

Stevo Stević [a,*], Sei-ichiro Ueki [b]

[a] Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia
[b] Faculty of Engineering, Ibaraki University, Hitachi 316-8511, Japan

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ABSTRACT

Let $\mathbb{B}$ denote the unit ball in $\mathbb{C}^n$ and $H(\mathbb{B})$ the space of all holomorphic functions on $\mathbb{B}$. We study the boundedness and compactness of the following integral-type operators

$$I_\varphi^g f(z) = \int_0^1 \Re f(\varphi(tz))g(tz)\frac{dt}{t}, \quad z \in \mathbb{B},$$

where $g \in H(\mathbb{B}), g(0) = 0, \varphi$ is a holomorphic self-map of $\mathbb{B}$ and $\Re f$ is the radial derivative of f, between weighted-type spaces on the unit ball.

1. Introduction

Let $\mathbb{B}$ denote the open unit ball of the n-dimensional complex vector space $\mathbb{C}^n$ and $H(\mathbb{B})$ denote the space of all holomorphic functions on $\mathbb{B}$.

For each $\alpha > 0$, we define the weighted-type space $H_\alpha^\infty(\mathbb{B})$ as follows:

$$H_\alpha^\infty(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \sup_{0<r<1} (1-r)^\alpha M_\infty(f,r) < \infty \right\},$$
$$M_\infty(f,r) = \sup_{|z|=r} |f(z)|.$$

It is easy to see that $f \in H_\alpha^\infty(\mathbb{B})$ if and only if $\sup_{z\in\mathbb{B}}(1-|z|)^\alpha|f(z)| < \infty$, so we define the norm $\|f\|_{H_\alpha^\infty}$ on $H_\alpha^\infty(\mathbb{B})$ by this supremum.

Furthermore we consider the subspace $H_{\alpha,0}^\infty(\mathbb{B})$ defined by

$$H_{\alpha,0}^\infty(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \lim_{r\to 1^-} (1-r)^\alpha M_\infty(f,r) = 0 \right\}.$$

Note that $H_{\alpha,0}^\infty(\mathbb{B})$ is a closed subspace of $H_\alpha^\infty(\mathbb{B})$. They are sometimes called the Bers-type spaces (see, e.g. [36,39]) and are special cases of the weighted-type space $H_\mu^\infty = H_\mu^\infty(\mathbb{B})$ (see, e.g. [25]). The properties of these spaces are discussed in [21]. For $f \in H(\mathbb{B})$ with the Taylor expansion $f(z) = \sum_{|\gamma|\geqslant 0} a_\gamma z^\gamma$, let

$$\Re f(z) = \sum_{|\gamma|\geqslant 0} |\gamma| a_\gamma z^\gamma,$$

* Corresponding author.
E-mail addresses: sstevic@ptt.rs (S. Stević), sei-ueki@mx.ibaraki.ac.jp (S.-i. Ueki).

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be the radial derivative of f, where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multi-index, $|\gamma| = \gamma_1 + \cdots + \gamma_n$ and $z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$. It is well-known that

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) = \langle \nabla f(z), \bar{z} \rangle,$$

where ∇ is the usual gradient on $\mathbb{C}^n$.

Assume that $g \in H(\mathbb{B}), g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$, then an integral-type operator, denoted by I_φ^g on $H(\mathbb{B})$, is defined as follows:

$$I_\varphi^g f(z) = \int_0^1 \Re f(\varphi(tz)) g(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), \quad z \in \mathbb{B}.$$

The operator I_φ^g was introduced and treated by Zhu and the first author of this paper in [30,32,38], where its boundedness and compactness from the Zygmund space, the mixed norm space and the generalized weighted Bergman space to the Bloch-type space on the unit ball are studied. Our motivation for introducing this operator stemmed from the operator I_φ which was introduced in a private communication by Li and Stević and was studied in [2,3,9–12,15,16,18]. For related integral-type operators beside just cited papers, see, also [1,5–8,13,14,17,19,20,22–24,26–29,31,33–35,37,40] as well as the related references therein.

In this paper we study the boundedness and compactness of the integral-type operator I_φ^g between different weighted-type spaces on the unit ball. We also give the estimate for its operator norm and/or essential norm.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \leqslant Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Auxiliary results

In this section, we quote several auxiliary results which are used in the proofs of the main results in Sections 3 and 4.

Lemma 1. *Assume that φ is a holomorphic self-map of $\mathbb{B}$ and $g \in H(\mathbb{B})$ with $g(0) = 0$. Then for every $f \in H(\mathbb{B})$ it holds*

$$\Re\left[I_\varphi^g f\right](z) = \Re f(\varphi(z)) g(z).$$

Proof. See, for example, [32, Lemma 3]. □

The following lemma was proved in [6, p. 2174, Theorem 2].

Lemma 2. *Let $\alpha > 0$ and m be a positive integer. Then for every $f \in H(\mathbb{B})$ it holds*

$$\sup_{0<r<1} (1-r)^\alpha M_\infty(f, r) \asymp |f(0)| + \sup_{0<r<1} (1-r)^{\alpha+m} M_\infty(\Re^m f, r).$$

Lemma 3. *Let $\alpha > 0$. Then for every $f \in H(\mathbb{B})$ it holds*

$$|\nabla f(z)| \preceq \frac{\|f\|_{H_\alpha^\infty}}{(1-|z|)^{\alpha+1}}.$$

Proof. By Lemma 2 and the following asymptotic relation from [4]

$$\sup_{z \in \mathbb{B}} (1-|z|)^{\alpha+1} |\nabla f(z)| \asymp \sup_{z \in \mathbb{B}} (1-|z|)^{\alpha+1} |\Re f(z)|,$$

we have

$$\|f\|_{H_\alpha^\infty} \succeq \sup_{z \in \mathbb{B}} (1-|z|)^{\alpha+1} |\Re f(z)| \succeq (1-|z|)^{\alpha+1} |\nabla f(z)|,$$

which proves the desired estimate. □

Lemma 4. *Let $\alpha > 0$ and $f \in H(\mathbb{B})$. If $(1-|z|)^\alpha |f(z)| \to 0$ as $|z| \to 1^-$, then $(1-|z|)^{\alpha+1} |\Re f(z)| \to 0$ as $|z| \to 1^-$.*

Proof. From the proof of [6, p. 2174, Theorem 2], we have

$$(1-r) M_\infty(\Re f, r) \preceq M_\infty\left(f, \frac{1+r}{2}\right),$$