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The submatrix constraint problem of matrix equation $AXB + CYD = E$ ☆

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ABSTRACT

We say that $X = [x_{ij}]_{i,j=1}^{n}$ is symmetric centrosymmetric if $x_{ij} = x_{ji}$ and $x_{n-j+1,n-i+1}$, $1 \leqslant i,j \leqslant n$. In this paper we present an efficient algorithm for minimizing $\|AXB + CYD - E\|$ where $\|\cdot\|$ is the Frobenius norm, $A \in \mathbb{R}^{t\times n}, B \in \mathbb{R}^{n\times s}, C \in \mathbb{R}^{t\times m}, D \in \mathbb{R}^{m\times s}, E \in \mathbb{R}^{t\times s}$ and $X \in \mathbb{R}^{n\times n}$ is symmetric centrosymmetric with a specified central submatrix $[x_{ij}]_{r\leqslant i,j\leqslant n-r}$, $Y \in \mathbb{R}^{m\times m}$ is symmetric with a specified central submatrix $[y_{ij}]_{1\leqslant i,j\leqslant p}$. Our algorithm produces suitable X and Y such that $AXB + CYD = E$ in finitely many steps, if such X and Y exists. We show that the algorithm is stable any case, and we give results of numerical experiments that support this claim.

1. Introduction

Let J_n be the $n \times n$ order flip matrix with ones on the secondary diagonal and zeros elsewhere. A matrix $X \in \mathbb{R}^{n\times n}$ is said to be symmetric centrosymmetric if $X^T = X$ and $J_n X J_n = X$, i.e.,

$$x_{j,i} = x_{i,j} = x_{n-j+1,n-i+1}, \quad 1 \leqslant i,j \leqslant n,$$

or symmetric centroskew if $X^T = X$ and $J_n X J_n = -X$, i.e.,

$$x_{j,i} = x_{i,j} = -x_{n-i+1,n-j+1}, \quad 1 \leqslant i,j \leqslant n.$$

Throughout this paper, we denote $\mathbb{S}^{n\times n}, \mathbb{AS}^{n\times n}, \mathbb{BS}^{n\times n}$ and $\mathbb{BAS}^{n\times n}$ are respectively the set of $n \times n$ real symmetric matrices, real skew-symmetric matrices, real symmetric centrosymmetric matrices and real symmetric skew-centrosymmetric. Let $\|Z\| = (\langle Z, Z\rangle)^{\frac{1}{2}}$ be the Frobenius matrix norm of a matrix $Z, \langle Z, Y\rangle = tr(Z^T Y)$ is the associated inner product of Z with a matrix Y, and $tr(W)$ denotes the trace of a square matrix W. By using properties of the trace operator, we have that for any matrices W, Y, and $Z, \langle W, YZ\rangle = \langle Y^T W, Z\rangle = \langle WZ^T, Y\rangle$. Denote $Y([1:p])$ by the p order leading principal submatrix of Y, i.e., $Y([1:p]) = [y_{i,j}]_{1\leqslant i,j\leqslant p}$. Denote

$$\mathscr{S}^{n\times n} \times \mathscr{S}^{m\times m} = \{[M,N] | M \in \mathscr{S}^{n\times n}, N \in \mathscr{S}^{m\times m}\}.$$

It is obvious that $\mathscr{S}^{n\times n} \times \mathscr{S}^{m\times m}$ is a linear subspace over the real number field.

Symmetric centrosymmetric matrices arises in many applications, for example, information theory [21], some Markov processes [22], physics and engineering problems [23], and have been extensively studied; see, e.g., [1–3, 22, 23]. Recently [4–6] there has been interest in the submatrix constraint problem of symmetric centrosymmetric matrices. However, because of the specified structure, it is unfit for discussing symmetric centrosymmetric matrices under their leading principal

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submatrices constraint, for they destroy the special doubly symmetry. Therefore, we present a concept of central principal submatrix, which was originally proposed by Yin [7]. The definition is as follows.

Definition 1. Given $M \in R^{n\times n}$, if $n-q$ is even, then a q-square central principal submatrix of A, denoted as $M_c(q)$, is a q-square submatrix obtained by deleting the first and last $(n-q)/2$ rows and columns of M, that is $M_c(q) = [m_{ij}]_{\frac{n-q}{2}\leqslant i,j\leqslant n-\frac{n-q}{2}}$.

It is intuitive and obvious that a matrix of odd (even) order only has central principal submatrices of odd (even) order.

The submatrix constraint problems are originally come from a practical subsystem expansion problem, and have been thoroughly investigated. For example, Deift and Nanda [8] discussed an inverse eigenvalue problem of a tridiagonal matrix under a submatrix constraint; Peng and Hu [9] considered an inverse eigenpair problem of a Jacobi matrix under a leading principal submatrix constraint; Gong [11] discussed antisymmetric solution of $AXA^T = B$ for X with a leading principal submatrix constraint. Zhao [10] studied bisymmetric solution of $AX = B$ for X with a central principal submatrix constraint. The solvability conditions, expressions of general solutions and optimal approximation solution are provided in these papers. However, to our best knowledge, there is no relative results about discussing simultaneously two different types of submatrices constraint associate with the well-known matrix equation [13,16–19]

$$AXB + CYD = E. \tag{1.1}$$

We also should point out that the arbitrary coefficient matrices A, B, C, D and E occurring in practice are usually obtained from experiments and they may not satisfy the solvability conditions. Therefore, we study the least squares problem. Thus, the problem can be mathematically formulated as follows.

Problem I. Let t, s, n, m, q, p be six positive integers. Let $A \in \mathbb{R}^{t\times n}$, $\mathbb{R} \in R^{n\times s}, C \in \mathbb{R}^{t\times m}, D \in \mathbb{R}^{m\times s}$, $E \in \mathbb{R}^{t\times s}$ and $X_0 \in \mathbb{BS}^{q\times q}, Y_0 \in \mathbb{S}^{p\times p}$. Let

$$\mathscr{S} = \{X | X \in \mathbb{BS}^{n\times n} \text{ with } X_c(q) = X_0\}, \quad \mathscr{T} = \{Y | Y \in \mathbb{S}^{m\times m} \text{ with } Y([1:p]) = Y_0\}. \tag{1.2}$$

Find matrix pair $[\widehat{X}, \widehat{Y}] \in \mathscr{S} \times \mathscr{T}$ such that

$$\|A\widehat{X}B + C\widehat{Y}D - E\| = \min_{[X,Y]\in\mathscr{S}\times\mathscr{T}} \|AXB + CYD - E\|. \tag{1.3}$$

We also consider the optimal approximation problem, which occurs frequently in structural identification [25].

Problem II. Let matrix pair $[X_*, Y_*] \in \mathscr{S} \times \mathscr{T}$ be given. Let S_E denote the solution set of Problem I, find matrix pair $[\widehat{X}_*, \widehat{Y}_*] \in S_E$ such that

$$\|\widehat{X}_* - X_*\|^2 + \|\widehat{Y}_* - Y_*\|^2 = \underset{[\widehat{X},\widehat{Y}]\in S_E}{n} \{\|\widehat{X} - X_*\|^2 + \|\widehat{Y} - Y_*\|^2\}.$$

Our results are natured extension of results obtain in [4–6,12]. These references describe application in which such problem size. In these papers, inevitably, Moore–Penrose generalized inverses and some complicated matrix decompositions such as canonical correlation decomposition (CCD) and general singular value decomposition (GSVD) are involved. Because of the obvious difficulties in numerical instability and computational complexity, those constructional solutions narrow down their applications. Indeed, it is impractical to find a solution by those formulas if the matrix size is large. In the present paper we extend and develop the above research, however, in a totally different way.

This paper we are only concerned with iteration method, and the main idea is based on the classical conjugate gradient least squares method (CGLS) [24] as well as the minimal residual iteration idea proposed in [14]. We first transform Problem I to an equivalent least squares problem over a linear subspace, it provides a way to construct an algorithm for solving the equivalent problem. With the proposed algorithm, the required submatrix constraint condition is automatically satisfied if the initial matrix pair is chosen within a certain set, and a solution can be obtained with finitely many steps. The algorithms require little work and low storage requirements per iteration. In fact, we need only to compute a residual matrix and update the iterative solution and gradient matrices *linearly* in each iteration. We have also verified the algorithm satisfies a minimization property, which ensures that this algorithm possesses a smooth convergence. In addition, the related optimal approximation problem is also solved. Some numerical results display the efficiency of these algorithms. Moreover, combined with numerical examples, we give some perturbation analysis on the approximation problem, and show that our algorithms is numerical stable associated with the approximation problem.

2. Preliminaries

We first make the following splitting of symmetric centrosymmetric matrix X into smaller submatrices

$$X = \begin{pmatrix} X_{11} & X_{12} & HJ_{\frac{n-q}{2}} \\ X_{12}^T & X_{22} & J_q X_{12}^T J_{\frac{n-q}{2}} \\ J_{\frac{n-q}{2}} H & J_{\frac{n-q}{2}} X_{12} J_q & J_{\frac{n-q}{2}} X_{11} J_{\frac{n-q}{2}} \end{pmatrix} \tag{2.1}$$

where $X_{11} \in \mathbb{S}^{\frac{n-q}{2}\times\frac{n-q}{2}}, X_{12} \in \mathbb{R}^{\frac{n-q}{2}\times q}, H \in \mathbb{S}^{\frac{n-q}{2}\times\frac{n-q}{2}}$, and $X_{22} \in \mathbb{BS}^{q\times q}$ (the q-square central principal submatrix of X). Actually, because of the symmetry properties of X, we can partition X into the following form: