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On a Steffensen–Hermite method of order three

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ABSTRACT

In this paper we study a third order Steffensen type method obtained by controlling the interpolation nodes in the Hermite inverse interpolation polynomial of degree 2. We study the convergence of the iterative method and we provide new convergence conditions which lead to bilateral approximations for the solution; it is known that the bilateral approximations have the advantage of offering a posteriori bounds of the errors. The numerical examples confirm the advantage of considering these error bounds.

1. Introduction

The iterative methods play a crucial role in approximating the solutions of nonlinear equations. The methods with superlinear convergence offer good approximations with a reduced number of steps. In a series of papers [1–11] the authors obtain different methods or modifications of some known methods, in order to achieve iterative methods with higher convergence orders.

The Steffensen, Aitken or Aitken–Steffensen methods lead to sequences having at least order 2 of convergence. A natural approach to generalize such methods can be obtained with the aid of inverse polynomial interpolation (Lagrange, Hermite, Taylor, etc.), with controlled interpolation nodes [12–17]. One of the advantages of such methods is the fact that the interpolation nodes may be controlled such that the methods offer sequences with bilateral approximations (both from above and from below) of the solutions. This aspect offers the control of the error at each step [14,16].

In this paper we shall extend a Steffensen type method using the Hermite inverse interpolatory polynomial of degree 2 with two nodes. In [13] we have shown that among all the Steffensen–Hermite methods with two nodes of arbitrary orders, the optimal efficiency index is attained in the case when one node is simple and the other one is double (see [18] for definitions of efficiency index); we have also shown there that the convergence order of this method is 3. Here we provide new convergence conditions, which offer bilateral approximations of the solution; these are very useful for controlling the error at each iteration step. In Section 2, we shall study the convergence of this method, and in Section 3 we shall indicate a method of constructing the auxiliary functions used for controlling the interpolations nodes. Some numerical examples will be shown in Section 4.

Let $c, d \in \mathbb{R}, c < d, f : [c, d] \to \mathbb{R}, g : [c, d] \to [c, d]$ and consider the following equivalent equations:

$$f(x) = 0, \tag{1}$$
$$g(x) = x. \tag{2}$$

As usually, the first order divided difference of f at $a, b \in [c, d]$ will be denoted by $[a, b; f]$; if a is double, then $[a, a; f] = f'(a)$. For the second order divided differences we have

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$$[a,b,e;f] = \frac{[b,e;f] - [a,b;f]}{e-a}, \quad a,b,e \in [c,d]$$

and if a is double, then

$$[a,a,b;f] = \frac{[a,b;f] - f'(a)}{b-a}.$$

Let $F = f([c,d])$ and assume the following conditions hold:

(**A**) $f \in C^3([c,d])$ and $f'(x) \neq 0 \ \forall x \in [c,d]$.

By **A** it follows that $f : [c,d] \to F$ is invertible so there exists $f^{-1} : F \to [c,d]$.

Let $a_i \in [c,d], i = 1,2$ and $b_i = f(a_i), i = 1,2,$ i.e. $a_i = f^{-1}(b_i)$, and denote $a'_1 = (f^{-1}(b_1))' = \frac{1}{f'(a_1)}$. Consider now the inverse interpolatory Hermite polynomial having b_1 as double node and b_2 as simple node, i.e. the second degree polynomial H determined such that

$$\begin{aligned} &H(b_1) = a_1, \\ &H'(b_1) = a'_1, \\ &H(b_2) = a_2. \end{aligned} \tag{3}$$

Using the divided differences on multiple nodes, the resulted Hermite polynomial may be expressed in one of the following equivalent ways [17]:

$$H(y) = a_1 + [b_1,b_2;f^{-1}](y-b_1) + [b_1,b_2,b_1;f^{-1}](y-b_1)(y-b_2), \tag{4}$$

$$H(y) = a_1 + [b_1,b_1;f^{-1}](y-b_1) + [b_1,b_1,b_2;f^{-1}](y-b_1)^2 \tag{5}$$

or

$$H(y) = a_2 + [b_2,b_1;f^{-1}](y-b_2) + [b_2,b_1,b_1;f^{-1}](y-b_2)(y-b_1). \tag{6}$$

The remainder is given by

$$f^{-1}(y) - H(y) = [y,b_1,b_1,b_2;f^{-1}](y-b_1)^2(y-b_2), \quad y \in F. \tag{7}$$

It can be easily seen that the representations given by (4)–(6) verify condition (3).

(**B**) Assume that Eq. (1) has a solution $\bar{x} \in [c,d]$.

By **A** it follows that the solution $\bar{x}$ is unique in $[c,d]$.

One has $\bar{x} = f^{-1}(0)$, whence, by (4)–(7), one obtains the following representations for $\bar{x}$:

$$\bar{x} = a_1 - [b_1,b_2;f^{-1}]b_1 + [b_1,b_2,b_1;f^{-1}]b_1b_2 - r, \tag{8}$$

$$\bar{x} = a_1 - [b_1,b_1;f^{-1}]b_1 + [b_1,b_1,b_2;f^{-1}]b_1^2 - r \tag{9}$$

or

$$\bar{x} = a_2 - [b_2,b_1;f^{-1}]b_2 + [b_2,b_1,b_1;f^{-1}]b_2b_1 - r, \tag{10}$$

where

$$r = [0,b_1,b_1,b_2;f^{-1}]b_1^2b_2. \tag{11}$$

If in (8), (9) or (10) we neglect the remainder r, one may obtain an approximation for $\bar{x}$, denoted by a_3:

$$a_3 = a_1 - [b_1,b_2;f^{-1}]b_1 + [b_1,b_2,b_1;f^{-1}]b_1b_2 \tag{12}$$

or

$$a_3 = a_1 - [b_1,b_1;f^{-1}]b_1 + [b_1,b_1,b_2;f^{-1}]b_1^2 \tag{13}$$

or

$$a_3 = a_2 - [b_1,b_2;f^{-1}]b_2 + [b_1,b_1,b_2;f^{-1}]b_1b_2. \tag{14}$$