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An effective linear approximation method for separable programming problems

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ABSTRACT

Among the numerous applications of piecewise linearization methods include data fitting, network analysis, logistics, and statistics. In the early 1950s, a concave function was found to be able to be linearized by introducing 0–1 variables. Most textbooks in Operations Research offer such methods for expressing linear approximations. Various methods of linearization have also been developed in recent literature. Nevertheless, the transformed linear scheme has a severe shortcoming: most standard procedures for linearizing typically involve a large increase in the number of binary variables. Consequently, the gains to be derived from dealing with linear functions are quite likely to be nullified by the increase in the size of the problem.

Conventional methods for linearizing a concave function with m break points require $m-1$ binary variables. However, when m becomes large, the computation will be very time-consuming and may cause a heavy computational burden.

This study proposes an effective approach in which only $\lceil \log_2(m-1) \rceil$ binary variables are used. The proposed method has the following features: (i) it offers more convenient and efficient means of expressing a piecewise linear function; (ii) fewer 0–1 variables are used; (iii) the computational results show that the proposed method is much more efficient and faster than the conventional one, especially when the number of break points becomes large.

1. Introduction

Optimization is applied in a variety of areas. Real systems are commonly described using mathematical models, whose parameters are sought to optimize the system in some ways. The search for these parameters is a global optimization problem.

In optimization, one distinguishes local and global optimization. Many efficient methods of local optimization are already available. Global optimization is more complex, and unlike local optimization, it can not be performed by relying on local information. Real problems typically have many local optima, of which the smallest is to be preferably determined. Hence, methods for global optimization are very computationally burdensome.

Classical methods compute results that can not be easily verified. There is no information on how closely the computed results approximate the "true" solution. The development of efficient methods for global optimization has made problems that used to have to be neglected some years ago because of the long solution time, solvable in an acceptable time today.

A natural approach for solving such problems is to approximate each general function by a piecewise linear one, and then reformulate the resulting problem as a discrete optimization problem. This reduction can often be performed in a way that

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preserves the structure of the problem, allowing existing discrete optimization techniques to be applied to the resulting problem. A wide variety of techniques is available for solving these problems, such as heuristic methods [1,2], integer programming methods [3,4], and approximation algorithms [5].

Piecewise linearization methods have numerous applications, including data fitting, network analysis, logistics, and statistics [6]. Nevertheless, the linear transformation approach often encounters a severe shortcoming. Standard procedures for linearizing typically involve a radical increase in the number of problem variables and constraints. Consequently, the gains made by dealing with linear functions are quite likely to be nullified by the increase in the size of the problem.

2. Literature review

The literature on piecewise linear approximation is extensive [7]. For the last five decades, piecewise linearization methods have been widely adopted to approximate concave functions. In the early 1950s, a concave function was determined to be able to be piecewisely linearized by introducing 0–1 variables. Most textbooks [6,8,9] in Operations Research provide such methods of expressing piecewise linear approximations. Many methods of piecewise linearization have also been recently developed. This section is limited to a brief survey of previous results on approximating concave functions in the context of optimization problems.

Sherali and Tuncbilek [10] proposed a reformulation-linearization technique (RLT) which generates polynomial constraints and then linearizes the resulting problem by defining new variables. This construction is then adopted to obtain lower bounds in the context of a branch-and-bound scheme. Later, Sherali et al. [11] developed a hierarchy of relaxations for solving mixed 0–1 integer problems as an extension of RLT. However, this algorithm is difficult to implement because numerous implied constraints have to be generated in a linearized form. Its representation at the expense of an exponential constraint is a long trial-and-error process. Moreover, it always requires the generation of huge amounts of bounded constraints.

Geoffrion [12] obtained various general results concerning the approximation of objective functions. He considered the minimization of a concave function over a general ground set. He derives conditions under which a piecewise linear approximation of the objective function achieves the smallest possible absolute error for a given number of pieces. Nevertheless, he did not set a bound on the number of pieces that were required to achieve a given precision.

Güder and Morris [13] studied the maximization of a concave function over a polyhedron. Güder and Morris approximated the objective function and then derived bounds on the number of segments needed to guarantee a given absolute error in terms of objective function and ground set parameters.

Hajiaghayi et al. [14] considered the unit-demand concave cost facility location problem and obtained an exact reduction by interpolating the concave functions at break points. The size of the resulting problem was a polynomial function of the size of the original problem, but this approach was limited to unit-demand problem.

Rosen and Pardolas [15] considered the minimization of a quadratic concave function over a polyhedron. They reduced the problem to a separable one and then approximated the resulting univariate concave functions. They specified a bound on the number of pieces required for a given precision in terms of objective function and ground set parameters.

Thakur [16] considered the maximization of a concave function over a convex set that was defined by separable constraints. He approximated both the objective and constraints functions, and bounded the number of pieces needed to guarantee a given absolute error as a function of problem parameters and the maximum values of the first and second derivatives of the given functions. However, none of these results bounded the number of pieces, and thus the size of the resulting instances was a polynomial in the size of the original instances.

3. Conventional piecewise linearization

Consider the following separable program (Problem **P**)
Problem **P**

$$\min \sum_{i=1}^{n} f_i(x_i) \tag{1}$$

$$s.t.$$

$$\sum_{j} b_{ij} x_i \leqslant p_j, \quad j = 1, 2, \ldots, l, \tag{2}$$

$$\underline{x}_i \leqslant x_i \leqslant \bar{x}_i, \quad i = 1, 2, \ldots, n. \tag{3}$$

where $b_{ij}, p_j, \underline{x}_i$, and $\bar{x}_i$ are constant, $f_i(x_i)$ is a function of a single variable x_i.

The linear approximation of the concave functions in problem **P** is now discussed. To simplify the expression, a continuous function $f(x)$ of a single variable x is considered, where x is within the interval $[a_1,\ a_m]$.

Consider a grid of m points throughout the interval of $[a_1, a_m]$ and denote the break points $a_1, a_2, \ldots, a_m$, where $a_1 < a_2 < \ldots < a_m$. $f(a_k)$ is evaluated at each a_k, and two adjacent break points $(a_k, f(a_k))$ and $(a_{k+1}, f(a_{k+1}))$ are connected with a straight line as shown in Fig. 1.