Applied Mathematics and Computation 215 (2009) 1573–1582

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Second-order nonlinear singular Sturm–Liouville problems with integral boundary conditions ☆

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ARTICLE INFO

Keywords:
Second-order singular differential equation
Positive solutions
Fixed point theory

ABSTRACT

This paper is concerned with the second-order singular Sturm–Liouville integral boundary value problems

$$\begin{cases} -u''(t) = \lambda h(t) f(t, u(t)), \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = \int_0^1 a(s)u(s)ds, \\ \gamma u(1) + \delta u'(1) = \int_0^1 b(s)u(s)ds, \end{cases}$$

where $\lambda > 0, h$ is allowed to be singular at $t = 0, 1$ and $f(t, x)$ may be singular at $x = 0$. By using the fixed point theory in cones, an explicit interval for λ is derived such that for any λ in this interval, the existence of at least one positive solution to the boundary value problem is guaranteed. Our results extend and improve many known results including singular and non-singular cases.

1. Introduction

In this paper, we study the existence of positive solutions for the following second-order nonlinear singular Sturm–Liouville integral boundary value problems (BVP)

$$\begin{cases} -u''(t) = \lambda h(t) f(t, u(t)), \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = \int_0^1 a(s)u(s)ds, \\ \gamma u(1) + \delta u'(1) = \int_0^1 b(s)u(s)ds, \end{cases} \tag{1.1}$$

where $\lambda > 0,\ \alpha, \beta, \delta, \gamma \geqslant 0$ are constants such that $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0,\ \ a, b \in C[0, 1]$ and positive, $f \in C([0, 1] \times (0, +\infty), [0, +\infty))$, $h(t)$ is allowed to be singular at $t = 0, 1$ and $f(t, x)$ may be singular at $x = 0$.

Boundary value problems for ordinary differential equations often arise from applied mathematics and physics, and the existence and multiplicity of positive solutions for such problems have become an important area of investigation in recent years. To mention a few, we refer the reader to [2,4–6,8–12,22–24] and the references therein. By using the Krasnoselskii fixed point theorem in a cone, Yao [1] proved the existence of at least one positive solution for the following second-order two-point boundary value problem

☆ The first and second authors are supported financially by the National Natural Science Foundation of China (10771117), the State Ministry of Education Doctoral Foundation of China (20060446001) and the Natural Science Foundation of Shandong Province of China (Y2007A23). The third author is supported financially by the Australia Research Council through an ARC Discovery Project Grant.

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doi:10.1016/j.amc.2009.07.024

$$\begin{cases} -u''(t) = h(t)f(u(t)), \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

where $h \in ((0,1),[0,+\infty)), f \in C((0,+\infty),[0,+\infty))$, $h(t)$ is allowed to be singular at both end $t = 0, t = 1$ and $f(x)$ is allowed to be singular at $x = 0$.

Ma [3] investigated the following three-point boundary value problem

$$\begin{cases} -u''(t) = h(t)f(u(t)), \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = \mu u(\eta), \end{cases}$$

where $0 < \eta < 1, 0 < \mu < 1, h \in C([0,1],[0,+\infty)), f \in C([0,+\infty),[0,+\infty))$ and there exists $t_0 \in [0,1]$ such that $h(t_0) > 0$. Under the assumption that the nonlinearity of f is either superlinear or sublinear, the existence of at least one positive solution was established by applying the Krasnoselskii fixed point theorem.

In a late work, by using the Leggett–Williams fixed point theorem, He and Ge [4] acquired the existence of triple positive solutions for

$$\begin{cases} -u''(t) = f(t,u(t)), \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = \alpha u(\eta), \end{cases}$$

where $0 < \eta < 1, 0 < \mu < 1/\eta$, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

More recently, by using the Avery five functional fixed point theorem, Zhang and Sun [9] obtained the existence of multiple positive solutions of the singular second-order m-point boundary value problem

$$-u''(t) = h(t)f(u(t)), \quad 0 < t < 1,$$

subject to the boundary conditions

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i),$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = 0,$$

$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i),$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u'(1) = 0,$$

respectively, where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, a_i \in [0,+\infty)$ and $h(t)$ may be singular at $t = 0$ and $t = 1$.

A class of boundary value problems with integral boundary conditions also arose from the study of problems in heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics. Such problems include two, three, multipoint and nonlocal boundary value problems as special cases. Moreover, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems and have attracted the attention of Khan [14], Gallardo [15], Karakostas and Tsamatos [16], Lomtatidze and Malaguti [17] and the references therein. For more information about the general theory of integral equations and their relation with boundary value problems, we refer the reader to the book of Corduneanu [18] and Agarwal and O'Regan [19].

Note that most of the above mentioned work was done on the assumptions that h or f has no singularity. Thus our purpose here is to study BVP (1.1) under some weaker conditions in which we not only allow h to have singularity at $t = 0, 1$, but also allow $f(t,x)$ to have singularity at $x = 0$. As far as we know, only very few works have been published [1,13] for the cases where $f(t,x)$ has singularity at $x = 0$. Moreover, in this paper it is possible to replace the Riemann integrals in the boundary conditions by Riemann–Stieltjes with minor modifications.

The rest of this paper is organized as follows. In Section 2, we present some lemmas that are used to prove our main results. In Section 3, the existence of positive solution for BVP (1.1) is established by using the fixed point theory in cone. For convenience of referencing, the fixed point theory is given below.

Let K be a cone in a Banach space E and let $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$, and $\overline{K}_{r,R} = \{x \in K : r \leqslant \|x\| \leqslant R\}$, where $0 < r < R < +\infty$.

Lemma 1.1 ([20,21]). *Let K be a positive cone in real Banach space E, $0 < r < R < +\infty$, and let $A : \overline{K}_{r,R} \to K$ be a completely continuous operator and such that*

(i) $\|Tx\| \leqslant \|x\|$ for $x \in \partial K_R$.
(ii) There exists $e \in \partial K_1$ such that $x \neq Tx + me$ for any $x \in \partial K_r$ and $m > 0$. Then A has a fixed point in $\overline{K}_{r,R}$.

Remark 1.1. If (i) and (ii) are satisfied for $x \in \partial K_r$ and $x \in \partial K_R$, respectively. Then Lemma 1.1 is still true.