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# Method of summation of some slowly convergent series

Paweł Woźny, Rafał Nowak *

*Institute of Computer Science, University of Wrocław, ul. Joliot-Curie 15, 50-383 Wrocław, Poland*

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ABSTRACT

A new method of summation of slowly convergent series is proposed. It may be successfully applied to the summation of generalized and basic hypergeometric series, as well as some classical orthogonal polynomial series expansions. In some special cases, our algorithm is equivalent to Wynn's epsilon algorithm, Weniger transformation [E.J. Weniger, Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series, Computer Physics Reports 10 (1989) 189–371] or the technique recently introduced by Čížek et al. [J. Čížek, J. Zamastil, L. Skála, New summation technique for rapidly divergent perturbation series. Hydrogen atom in magnetic field, Journal of Mathematical Physics 44 (3) (2003) 962–968]. In the case of trigonometric series, our method is very similar to the Homeier's $\mathscr{H}$ transformation, while in the case of orthogonal series — to the $\mathscr{K}$ transformation. Two iterated methods related to the proposed method are considered. Some theoretical results and several illustrative numerical examples are given.



## 1. Introduction

Many fields of numerical mathematics come up against the convergence problem. In the case of slow convergence, many methods of acceleration were proposed. But still the development of the new algorithms plays an important role. In this paper, a new method of summation of slowly convergent series is introduced, and its efficiency is examined by many numerical examples. The method is a Levin-type sequence transformation; its relations to the classic methods are analysed.

For the purpose of this paper, let us consider an infinite series

$$s = \sum_{n=0}^{\infty} a_n$$

with terms $a_n$, partial sums $s_n = \sum_{j=0}^{n-1} a_j$, and remainders $r_n = \sum_{j=n}^{\infty} a_j$. Let the linear difference operator $\mathbb{L}^{(\infty)}$ annihilate the remainder $r_n$, i.e. $\mathbb{L}^{(\infty)}(r_n) = 0$. Throughout the paper, we assume that all difference operators act on the variable $n$. Observe that

$$s = \frac{\mathbb{L}^{(\infty)}(s_n)}{\mathbb{L}^{(\infty)}(1)}$$

provided $\mathbb{L}^{(\infty)}(1) \neq 0$. In general, it is difficult to obtain explicit form of the annihilator $\mathbb{L}^{(\infty)}$. Let the linear difference operator $\mathbb{L}^{(m)}, m \in \mathbb{N}$, be an approximation of $\mathbb{L}^{(\infty)}$ in the sense that

$$\begin{cases} \mathbb{L}^{(m)}(r_n^{(m)}) = 0, \\ \mathbb{L}^{(m)}(1) \neq 0, \end{cases}$$

* Corresponding author.
*E-mail addresses:* pwo@ii.uni.wroc.pl (P. Woźny), rno@ii.uni.wroc.pl (R. Nowak).

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where $r_n^{(m)} := r_n - r_{n+m}$. Then, we can expect that $\mathbb{L}^{(m)}(s_n)/\mathbb{L}^{(m)}(1)$ gives an approximation of $s$, of accuracy growing when $m \to \infty$.

In this paper, we consider the $\mathcal{Q}$ transformation, where $\mathcal{Q}^{(m)} : \{s_n\} \mapsto \{\mathcal{Q}_n^{(m)}\}, m \in \mathbb{N}$, is defined by

$$\mathcal{Q}_n^{(m)} := \frac{\mathbb{L}^{(m)}(s_n)}{\mathbb{L}^{(m)}(1)}.$$

Since $\mathbb{L}^{(m)}$ is a linear difference operator, $\mathcal{Q}$ is a Levin-type sequence transformation; see, e.g., [1], [3, Section 2.7], [4] or [5]. Different Levin-type transformations, and hence different methods of convergence acceleration, were studied by Weniger [1], Homeier [4], Brezinski and Matos [6], and Matos [7].

In Section 2, we give a method of obtaining the annihilators $\mathbb{L}^{(m)}$, satisfying $\mathbb{L}^{(m)} = \mathbb{P}^{(m)}\mathbb{L}^{(m-1)}$ for some linear difference operators $\mathbb{P}^{(m)}, m \in \mathbb{N}$. Thus we will consider

$$\mathbb{L}^{(m)} = \mathbb{P}^{(m)}\mathbb{P}^{(m-1)} \cdots \mathbb{P}^{(1)}$$

and we will say that $\mathcal{Q}$ is *determined* by operators $\mathbb{P}^{(m)}$. Next, we will make use of the fact that numerators and denominators on the right side of (2.3) satisfy the recurrence relation

$$X_n^{\{(m)\}} = \{\mathbb{P}\}^{\{(m)\}}(X_n^{\{(m-1)\}})$$

to propose an efficient algorithm of computing $\mathcal{Q}_n^{(m)}$. One could obtain explicit forms of the operators $\mathbb{P}^{(m)}, m \in \mathbb{N}$, provided the analytical form of terms $a_n$ is known. In the sequel, we give some applications of the $\mathcal{Q}$ transformation. In Section 3, we give explicit forms of the operators $\mathbb{P}^{(m)}$ in the case of generalized hypergeometric series, while in Section 4 we give analogous results for basic hypergeometric series. In Section 5, we show that our method may be successfully applied to the summation of some classical orthogonal polynomial series, too. Two iterated methods, related to the $\mathcal{Q}$ transformation, are considered in Section 6.

In the sequel, we will use the following notation:

- *shift operator*: $\mathbb{E}x_n = x_{n+1}$; more generally $\mathbb{E}^k x_n = x_{n+k}, k \in \mathbb{Z}$;
- *identity operator*: $\mathbb{I} := \mathbb{E}^0$;
- *linear difference operator*:

$$\mathbb{K} = \sum_{j=u}^{v} \lambda_j(n)\mathbb{E}^j \quad (\lambda_u(n) \not\equiv 0,\ \lambda_v(n) \not\equiv 0) \quad \text{of } \textit{order} \operatorname{ord}\mathbb{K} := v - u; \tag{1.1}$$

- *forward difference operator*: $\Delta := \mathbb{E} - \mathbb{I}$.

All computations and numerical tests are made in Maple™11 system, using 128-digit arithmetic. Following Paszkowski [8], we measure the *accuracy* of approximation $\sigma$ by

$$\mathtt{acc}(\sigma) := -\log_{10}\left|\frac{\sigma}{s} - 1\right|,$$

i.e., by the number of exact significant decimal digits in approximation $\sigma$ of the sum $s$.

## 2. The method

Consider the series

$$s = \sum_{n=0}^{\infty} a_n \tag{2.1}$$

with terms $a_n$, partial sums $s_n = \sum_{j=0}^{n-1} a_j$, and remainders $r_n = \sum_{j=n}^{\infty} a_j$. In Section 2.1, we propose a method of obtaining the annihilators $\mathbb{L}^{(m)}, m \in \mathbb{N}$, satisfying

$$\mathbb{L}^{(m)}(r_n^{(m)}) = 0, \tag{2.2}$$

where

$$r_n^{(m)} = r_n - r_{n+m} = a_n + a_{n+1} + \cdots + a_{n+m-1}.$$

We consider the transformation $\mathcal{Q}^{(m)} : \{s_n\} \mapsto \{\mathcal{Q}_n^{(m)}\}$, defined by

$$\mathcal{Q}_n^{(m)} := \frac{\mathbb{L}^{(m)}(s_n)}{\mathbb{L}^{(m)}(1)}, \quad m \in \mathbb{N}. \tag{2.3}$$

In Section 2.2, we describe an algorithm of computing $\mathcal{Q}_n^{(m)}$.