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Successive matrix squaring algorithm for computing outer inverses

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ABSTRACT

In this paper, we derive a successive matrix squaring (SMS) algorithm to approximate an outer generalized inverse with prescribed range and null space of a given matrix $A \in \mathbb{C}_r^{m\times n}$. We generalize the results from the papers [L. Chen, E.V. Krishnamurthy, I. Macleod, Generalized matrix inversion and rank computation by successive matrix powering, Parallel Computing 20 (1994) 297–311; Y. Wei, Successive matrix squaring algorithm for computing Drazin inverse, Appl. Math. Comput. 108 (2000) 67–75; Y. Wei, H. Wu, J. Wei, Successive matrix squaring algorithm for parallel computing the weighted generalized inverse $A_{MN}^{\dagger}$, Appl. Math. Comput. 116 (2000) 289–296], and obtain an algorithm for computing various classes of outer generalized inverses of A. Instead of particular matrices used in these articles, we use an appropriate matrix $R \in \mathbb{C}_s^{n\times m}$, $s \leqslant r$. Numerical examples are presented.

1. Introduction and preliminaries

Let $\mathbb{C}^{m\times n}$ and $\mathbb{C}_r^{m\times n}$ denote the set of all complex $m \times n$ matrices and all complex $m \times n$ matrices of rank r, respectively. I_n denotes the unit matrix of order n. By A^*, $\mathcal{R}(A)$, $\mathrm{rank}(A)$ and $\mathcal{N}(A)$ we denote the conjugate transpose, the range, the rank and the null space of $A \in \mathbb{C}^{m\times n}$. By $\mathrm{Re}z$ and $\mathrm{Im}z$ we denote a real and imaginary part of a complex number z, respectively.

For $A \in \mathbb{C}^{m\times n}$, the set of inner and outer generalized inverses are defined by the following, respectively:

$$A\{1\} = \{X \in \mathbb{C}^{n\times m} | AXA = A\}, \quad A\{2\} = \{X \in \mathbb{C}^{n\times m} | XAX = X\}.$$

The set of all outer inverses with prescribed rank s is denoted by $A\{2\}_s$, $0 \leqslant s \leqslant r = \mathrm{rank}(A)$. The symbols A^- or $A^{(1)}$ stand for an arbitrary generalized inner inverse of A and by $A^{(2)}$ we denote an arbitrary generalized outer inverse of A. Also, the matrix X which satisfies

$$AXA = A \quad \text{and} \quad XAX = X$$

is called the reflexive g-inverse of A and it is denoted by $A^{(1,2)}$. The set of all reflexive g-inverses is denoted by $A\{1,2\}$. Subsequently, the sets of $\{1,2,3\}$ and $\{1,2,4\}$ inverses of A are defined by

$$A\{1,2,3\} = A\{1,2\} \cap \{X | (AX)^* = AX\},$$
$$A\{1,2,4\} = A\{1,2\} \cap \{X | (XA)^* = XA\}.$$

By $A^{\dagger}$ we denote the Moore–Penrose inverse of A, i.e. the unique matrix $A^{\dagger}$ satisfying

$$AA^{\dagger}A = A, \quad A^{\dagger}AA^{\dagger} = A^{\dagger}, \quad (AA^{\dagger})^* = AA^{\dagger}, \quad (A^{\dagger}A)^* = A^{\dagger}A.$$

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For $A \in \mathbb{C}^{n\times n}$ the smallest nonnegative integer k such that $\mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)$ is called the index of A and denoted by $\mathrm{ind}(A)$. If $A \in \mathbb{C}^{n\times n}$ is a square matrix with $\mathrm{ind}(A) = k$, then the matrix $X \in \mathbb{C}^{n\times n}$ which satisfies the following conditions

$$A^kXA = A^k, \quad XAX = X, \quad AX = XA$$

is called the Drazin inverse of A and it is denoted by A^D. When $\mathrm{ind}(A) = 1$, Drazin inverse A^D is called the group inverse and it is denoted by $A^{\#}$.

Suppose that M and N are Hermite positive definite matrices of the order m and n, respectively. Then there exists the unique matrix $X \in \mathbb{C}^{n\times m}$ such that

$$AXA = A, \quad XAX = X, \quad (MAX)^* = MAX, \quad (NXA)^* = NXA.$$

The matrix X is called the weighted Moore–Penrose inverse of A, and denoted by $X = A^{\dagger}_{M,N}$. In particular, if $M = I_m$ and $N = I_n$, then $A^{\dagger}_{M,N} = A^{\dagger}$.

If $A \in \mathbb{C}^{n\times m}$ and $W \in \mathbb{C}^{m\times n}$, then the unique solution $X \in \mathbb{C}^{n\times m}$ of the equations

$$(AW)^{k+1}XW = (AW)^k, \quad XWAWX = X, \quad AWX = XWA, \tag{1.1}$$

where $k = \mathrm{ind}(AW)$, is called the W-weighted Drazin inverse of A and it is denoted by $A^{D,W}$.

If $A \in \mathbb{C}^{m\times n}_r$, T is a subspace of $\mathbb{C}^n$ of dimension $t \leqslant r$ and S is a subspace of $\mathbb{C}^m$ of dimension $m - t$, then A has a $\{2\}$ inverse X such that $\mathscr{R}(X) = V$ and $\mathscr{N}(X) = U$ if and only if

$$AV \oplus U = \mathbb{C}^m$$

in which case X is unique and we denote it by $A^{(2)}_{V,U}$.

It is well-known that for $A \in \mathbb{C}^{m\times n}$, the Moore–Penrose $A^{\dagger}$, the weighted Moore–Penrose inverse $A^{\dagger}_{M,N}$ and the weighted Drazin inverse $A^{D,W}$ can be represented by:

(i) $A^{\dagger} = A^{(2)}_{\mathscr{R}(A^*),\mathscr{N}(A^*)}$,
(ii) $A^{\dagger}_{M,N} = A^{(2)}_{\mathscr{R}(A^{\sharp}),\mathscr{N}(A^{\sharp})}$, where $A^{\sharp} = N^{-1}A^*M$,
(iii) $A^{D,W} = (WAW)^{(2)}_{\mathscr{R}(A(WA)^k),\mathscr{N}(A(WA)^k)}$, where $W \in \mathbb{C}^{n\times n}, k = \mathrm{ind}(WA)$.

Also, for $A \in \mathbb{C}^{n\times n}$, the Drazin inverse A^D can be represented by:

$$A^D = A^{(2)}_{\mathscr{R}(A^k),\mathscr{N}(A^k)}, \quad \text{where } \mathrm{ind}(A) = k.$$

The following representations of {2, 3}, {2, 4}-inverses with prescribed rank s are restated from [11]:

Proposition 1.1. *Let $A \in \mathbb{C}^{m\times n}_r$ and $0 < s < r$ be a chosen integer. Then the following is valid:*

(a) $A\{2,4\}_s = \{(ZA)^{\dagger}Z | Z \in \mathbb{C}^{s\times m}, ZA \in \mathbb{C}^{s\times n}_s\}$.
(b) $A\{2,3\}_s = \{Y(AY)^{\dagger} | Y \in \mathbb{C}^{n\times s}, AY \in \mathbb{C}^{m\times s}_s\}$.

General representations for various classes of generalized inverses can be found in [4,8,10,12]. Some of these representations are restated here for the sake of completeness.

Proposition 1.2. *Let $A \in \mathbb{C}^{m\times n}_r$ be an arbitrary matrix and $A = PQ$ is a full-rank factorization of A. There are the following general representations for some classes of generalized inverses:*

$$A\{2\}_s = \{F(GAF)^{-1}G | F \in \mathbb{C}^{n\times s}, G \in \mathbb{C}^{s\times m}, \mathrm{rank}(GAF) = s\},$$
$$A\{2\} = \bigcup_{s=0}^{r} A\{2\}_s,$$
$$A\{1,2\} = \{F(GAF)^{-1}G | F \in \mathbb{C}^{n\times r}, G \in \mathbb{C}^{r\times m}, \mathrm{rank}(GAF) = r\} = A\{2\}_r,$$
$$A\{1,2,3\} = \{F(P^*AF)^{-1}P^* | F \in \mathbb{C}^{n\times r}, \mathrm{rank}(P^*AF) = r\},$$
$$A\{1,2,4\} = \{Q^*(GAQ^*)^{-1}G | G \in \mathbb{C}^{r\times m}, \mathrm{rank}(GAQ^*) = r\},$$
$$A^{\dagger} = Q^*(P^*AQ^*)^{-1}P^*,$$
$$A^D = P_{A^l}(Q_{A^l}AP_{A^l})^{-1}Q_{A^l}, \quad A^l = P_{A^l}Q_{A^l}, \quad l \geqslant \mathrm{ind}(A).$$

For other important properties of generalized inverses see [1,2,6,13]. We will use the following well-known result:

Lemma 1.1 [7]. *Let $M \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$ be given. There is at least one matrix norm $\|\cdot\|$ such that*

$$\rho(M) \leqslant \|M\| \leqslant \rho(M) + \epsilon, \tag{1.2}$$

where $\rho(M)$ denotes the spectral radius of M.