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The modified KdV equation with variable coefficients: Exact uni/bi-variable travelling wave-like solutions

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ABSTRACT

In this paper, the modified Korteweg–de Vries (mKdV) equation with variable coefficients (vc-mKdV equation) is investigated via two kinds of approaches and symbolic computation. On the one hand, we firstly reduce the vc-mKdV equation to a second-order nonlinear nonhomogeneous ODE using travelling wave-like similarity transformation. And then we obtain its many types of exact fractional solutions with one travelling wave-like variable by applying some fractional transformations to the obtained nonlinear ODE. On the other hand, we reduce the vc-mKdV equation to two nonlinear PDEs with variable coefficients using the anti-tangent and anti-hypertangent function transformations, respectively. And then we given its many types of exact solutions with two different travelling wave-like variables by studying the obtained nonlinear PDE with variable coefficients.

1. Introduction

The modified Korteweg–de Vries (mKdV) equation [1–5]

$$u_t + 6\mu u^2 u_x + u_{xxx} = 0, \quad \mu = \pm 1 \tag{1.1}$$

is of important significance in many branches of nonlinear science field. When $\mu = 1$, (1.1) is called the positive mKdV equation, while $\mu = -1$, (1.1) is called the negative mKdV equation. The well-known Miura transformation [6]: $v = u_x + u^2$ becomes a bridge between (1.1) with $\mu = -1$ and the KdV equation: $v_t - 6vv_x + v_{xxx} = 0$. The mKdV equation appears in many fields such as acoustic waves in certain anharmonic lattices [7], Alfvén waves in a collisionless plasma [8], transmission lines in Schottky barrier [9], models of traffic congestion [10], ion acoustic solitons [11], elastic media [12], etc. It possesses many remarkable properties such as Miura transformation, conservation laws, inverse scattering transformation, bilinear transformation, *N*-solitons, breather solutions, Bäcklund transformation, Painlevé integrability, Darboux transformation, doubly periodic solutions, etc. [1–19].

It is also important to study the nonlinear wave equations with variable coefficients. More recently, Pradhan and Panigrahi [20] studied the modified KdV equation with variable coefficients

$$u_t + \alpha(t)u_x - \beta(t)u^2 u_x + \gamma(t)u_{xxx} = 0, \tag{1.2}$$

and some Jacobi elliptic function solutions with the forms $A\,\mathrm{sn}(\xi, m)$, $B\,\mathrm{cn}(\xi, m)$, $C\,\mathrm{dn}(\xi, m)$ were obtained by reducing (1.2) to one second-order ODE in the form $g''(\omega) = Pg(\omega) + 2Qg^3(\omega)$.

In this paper, we will investigate more types of solutions of (1.2) using some powerful transformations. In Section 2, we firstly reduce (1.2) to one second-order nonlinear ODE and then obtain some fractional solutions with one travelling wave-like variable.

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In Section 3, we give some exact breather-like and doubly periodic wave-like solutions with two different travelling wave-like variables using the anti-tangent and anti-hypertangent function transformations and other transformations.

2. Uni-variable travelling wave-like solutions

In this section, we will seek the solution with one travelling wave-like variable $\xi(x,t)$:

$$u(x,t) = A(t)w[\xi(x,t)] + B(t), \quad \xi(x,t) = f(t)x + g(t), \tag{2.1}$$

where $A(t) \not\equiv 0$, $B(t)$, $f(t)$, $g(t)$ are functions of t to be determined and $w(\xi)$ is a function of ξ. To determine these functions, we require that the function $w(\xi(x,t))$ satisfies the second-order nonlinear ODE

$$w''[\xi(x,t)] = \lambda w[\xi(x,t)] + kw^3[\xi(x,t)] + c, \tag{2.2}$$

where λ, k, c are constants. The substitution of (2.1) into (1.2) along with (2.2) yields a polynomial equation in $x^i w^j w'^s$ $(i,s = 0,1; j = 0,1,2)$. Setting their coefficients to zero leads to the following set of nonlinear ordinary differential equations:

$$\begin{cases} A(t)f'(t) = 0, \quad B'(t) = 0, \quad A'(t) = 0, \\ -2A^2(t)B(t)\beta(t)f(t) = 0, \\ 3kA(t)\gamma(t)f^3(t) - A^3(t)\beta(t)f(t) = 0, \\ A(t)g'(t) + A(t)\alpha(t)f(t) - A(t)B^2(t)\beta(t)f(t) + \lambda A(t)\gamma(t)f^3(t) = 0, \end{cases}$$

from which we have

$$\begin{aligned} &A(t) = A = \text{const}, \quad B(t) = 0, \quad \frac{\beta(t)}{\gamma(t)} = \mu = \text{const}, \\ &f(t) = A\sqrt{\frac{\mu}{3k}}, \quad g(t) = -A\sqrt{\frac{\mu}{3k}}\int^t \left[\alpha(s) + \frac{\lambda\mu A^2}{3k}\gamma(s)\right] ds, \end{aligned} \tag{2.3}$$

Case 1. $c = 0$.

In this case, (2.2) reduces to the form:

$$w''[\xi(x,t)] = \lambda w[\xi(x,t)] + kw^3[\xi(x,t)], \tag{2.4}$$

from which we have the equivalent form of (2.4)

$$\int \frac{dw[\xi(x,t)]}{\sqrt{\lambda w^2[\xi(x,t)] + \frac{1}{2}kw^4[\xi(x,t)] + \xi_0}} = \xi(x,t), \quad \xi_0 = \text{const}, \tag{2.4$'$}$$

from whose solutions, Pradhan and Panigrahi [20] had gave some Jaocibi elliptic functions of (1.2). In fact, (2.4) has also other types of solutions [16,21]. Here we do not consider this case.

Case 2. $c \neq 0$.

In this case, (2.2) is so different from (2.4). By choosing the proper parameters λ, k and c, we investigate some types of solutions of (2.2) using some transformations [22–24] such that the corresponding fractional travelling wave-like solutions of (1.2) are given by the following families:

Family 1 (*Rational wave-like solution*). Suppose that (2.2) has the solution $w(\xi(x,t)) = \frac{a+b\xi^2(x,t)}{d+\xi^2(x,t)}$, where a, b, d are constants to be determined. We substitute this expression into (2.2) and balance the coefficients of $\xi^i(x,t)$ to yields a set of algebraic equations such that these parameters can be determined by solving the set of equations. Therefore from (2.1) and (2.3) we get the solution of (1.2):

$$u_1(x,t) = \sqrt{-\frac{k}{3\lambda}}\frac{-9\lambda + 2\lambda^2\left\{A\sqrt{\frac{\mu}{3k}}x - A\sqrt{\frac{\mu}{3k}}\int^t\left[\alpha(s) + \frac{\lambda\mu A^2}{3k}\gamma(s)\right]ds\right\}^2}{3k + 2k\lambda\left\{A\sqrt{\frac{\mu}{3k}}x - A\sqrt{\frac{\mu}{3k}}\int^t\left[\alpha(s) + \frac{\lambda\mu A^2}{3k}\gamma(s)\right]ds\right\}^2}, \tag{2.5}$$

Family 2 (*Periodic wave-like solutions*). Suppose that (2.2) has form solution $w(\xi(x,t)) = \frac{a+b\sin^2(\xi(x,t))}{d+\sin^2(\xi(x,t))}$. Similarly, we get the solution of (1.2):

$$u_2(x,t) = \frac{-bd(2d+3) + b(2d+1)\sin^2\left\{A\sqrt{\frac{\mu}{3k}}x - A\sqrt{\frac{\mu}{3k}}\int^t\left[\alpha(s) + \frac{\lambda\mu A^2}{3k}\gamma(s)\right]ds\right\}}{(2d+1)d + (2d+1)\sin^2\left\{A\sqrt{\frac{\mu}{3k}}x - A\sqrt{\frac{\mu}{3k}}\int^t\left[\alpha(s) + \frac{\lambda\mu A^2}{3k}\gamma(s)\right]ds\right\}}, \tag{2.6}$$

where $\lambda = -\frac{4d^2+4d+3}{2d(d+1)}$, $k = -\frac{(2d+1)^2}{2db^2(d+1)}$.