Applied Mathematics and Computation 203 (2008) 210–216

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On a class of analytic multivalent functions

Janusz Sokół

Department of Mathematics, Rzeszów University of Technology, ul. W.Pola 2, PL-35-959 Rzeszów, Poland

ARTICLE INFO

Keywords:
Dziok–Srivastava operator
Bernardi operator
Hadamard product (or convolution)
Subordination
Starlike functions
Convex functions

ABSTRACT

We use a property of the Bernardi operator in the theory of the Briot–Bouquet differential subordinations to prove several theorems for the classes $V_k^p(\mathscr{H};A,B)$ of multivalent analytic functions defined by using the Dziok–Srivastava operator $\mathscr{H}$. Some of these results we obtain applying the convolution property due to Rusheweyh. We take advantage of the Miller–Mocanu lemma to improve the earlier result.

© 2008 Elsevier Inc. All rights reserved.

1. Introduction

Let $\mathscr{A}_{p,k}$ denote the class of functions f of the form:

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad (p < k; p,k \in \mathbf{N} = \{1,2,\ldots\}), \tag{1}$$

which are analytic in $\mathscr{U} = \mathscr{U}(1)$, where $\mathscr{U}(r) = \{z : z \in \mathbf{C} \text{ and } |z| < r\}$. Also let us put $\mathscr{A} = \mathscr{A}_{1,2}$. For analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n (z \in \mathscr{U})$$

by $f * g$ we denote the Hadamard product or convolution of f and g, defined by

$$(f^* g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in $\mathscr{U}$ such that

$$\omega(0) = 0, \quad |\omega(z)| < 1 (z \in \mathscr{U})$$

and

$$f(z) = g(\omega(z))(z \in \mathscr{U}).$$

In particular, if g is univalent in $\mathscr{U}$, we have the following equivalence:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathscr{U}) \subseteq g(\mathscr{U}).$$

E-mail address: jsokol@prz.edu.pl

0096-3003/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2008.04.027

A set $E \subset \mathbf{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E. A set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. Let f be analytic and univalent in $\mathscr{U}$. Then f maps $\mathscr{U}$ onto a convex domain if and only if

$$\mathrm{Re}\left[1+\frac{zf''(z)}{f'(z)}\right] > 0 \quad \text{in } \mathscr{U}. \tag{2}$$

Such function f is said to be convex in $\mathscr{U}$ (or briefly convex). The condition (2) for convexity was first stated by Study [24]. Now let $f(0) = 0$ and let f be analytic univalent in $\mathscr{U}$. Then f maps $\mathscr{U}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \quad \text{in } \mathscr{U}. \tag{3}$$

Such function f is said to be starlike in $\mathscr{U}$ with respect to $w_0 = 0$ (or briefly starlike). The condition (3) for starlikeness is due to Nevalinna [15]. It is well known that if an analytic function f satisfies (3) and $f(0) = 0, f'(0) \neq 0$, then f is univalent and starlike in $\mathscr{U}$.

One can alter the conditions (2) and (3) by setting other limitations on the behaviour of $zf'(z)/f(z)$ and of $zf''(z)/f'(z)$ in $\mathscr{U}$. In this way many interesting classes of analytic functions have been defined (see for instance [7]). Robertson introduced in [17] the classes $\mathscr{S}^*(\alpha)$, $\mathscr{K}(\alpha)$ of starlike and convex functions of order $\alpha \leqslant 1$, which are defined by

$$\begin{aligned} \mathscr{S}^*(\alpha) &:= \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z},\ z \in \mathscr{U} \right\} \\ &= \left\{ f \in \mathscr{A} : \mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha,\ z \in \mathscr{U} \right\}, \\ \mathscr{K}^*(\alpha) &:= \left\{ f \in \mathscr{A} :\ zf'(z) \in \mathscr{S}^*(\alpha) \right\}. \end{aligned} \tag{4}$$

If $\alpha \in [0;1)$, then a function in either of these sets is univalent, if $\alpha < 0$ it may fail to be univalent. In particular we denote $\mathscr{S}^*(0) = \mathscr{S}^*, \mathscr{K}(0) = \mathscr{K}$.

Janowski [8] introduced the class

$$\mathscr{S}^*\left[\frac{1+Az}{1+Bz}\right] := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, z \in \mathscr{U} \right\} \quad (-1 \leqslant B < A \leqslant 1). \tag{5}$$

In this paper we take advantage of $\mathscr{S}^*[\frac{1+Az}{1+Bz}]$ to define other class of functions.

Let $q, s \in \mathbf{N}_0 = \mathbf{N} \cup \{0\},\ q \leqslant s+1$. For complex parameters $a_1, \ldots, a_q$ and $b_1, \ldots, b_s$, $(b_j \neq 0, -1, -2, \ldots;\ j = 1, \ldots, s)$, the generalized hypergeometric function ${}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z)$ is defined by

$${}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \cdots (b_s)_n} \frac{z^n}{n!} \quad (z \in \mathscr{U}), \tag{6}$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & (n = 0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbf{N}). \end{cases}$$

Definition 1. Let $\mathscr{H} : \mathscr{A}_{p,k} \to \mathscr{A}_{p,k}$ be a operator such that

$$\mathscr{H}f(z) = \mathscr{H}(a_1, \ldots, a_q; b_1, \ldots, b_s)f(z) = [z^p \cdot {}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z)] * f(z),$$

where ${}_qF_s$ is given by (6).

This operator is called the Dziok–Srivastava operator [6]. We observe that for a function f of the form (1), we have

$$\mathscr{H}(a_1, \ldots, a_q; b_1, \ldots, b_s)f(z) = z^p + \sum_{n=k}^{\infty} A_n a_n z^n, \tag{7}$$

where

$$A_n = \frac{(a_1)_{n-p} \cdots (a_q)_{n-p}}{(b_1)_{n-p} \cdots (b_s)_{n-p} \cdot (n-p)!}.$$

The Dziok–Srivastava operator $\mathscr{H}(a_1, \ldots, a_q; b_1, \ldots, b_s)$ includes various other linear operators which were considered in earlier works. Now we show a few of them. For $p = s = 1$ and $q = 2$ and $a_2 = 1$, the Dziok–Srivastava operator becomes the Carlson–Shaffer operator $\mathscr{L}$:

$$\mathscr{L}(a_1, b_1)f(z) = \mathscr{H}(a_1, 1; b_1)f(z), \tag{8}$$

Download English Version:

https://daneshyari.com/en/article/4633382

Download Persian Version:

https://daneshyari.com/article/4633382

Daneshyari.com