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A symmetric linear system solver

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ABSTRACT

This paper presents a matrix factorization called WW^{T} factorization for the solution of the linear system of equations $Ax = f$, where the coefficient matrix A is symmetric positive definite. The existence theorems are proved and the backward error analysis of the method is given. A square root free WW^{T} factorization called WDW^{T} factorization is also presented.

1. Introduction

Systems of simultaneous linear equations occur in solving problems in a wide variety of disciplines, including mathematics, statistics, the physical, biological, and social sciences, engineering and business. They arise directly in solving real-world problems, and they also occur as part of the solution process for other problems, for example, solving system of simultaneous non-linear equations. Because of widespread importance of linear equations, much research has been devoted to their numerical solution (see [1]).

The problem of solving a system of linear equations

$$Ax = f \tag{1.1}$$

is central to the field of matrix computations. When A is a symmetric positive definite matrix, it is possible to factor A in the form $A = LL^{\mathrm{T}}$ for some lower triangular matrix L. This is known as Cholesky factorization. A variant of the classical Cholesky factorization, called Cholesky QIF (Quadrant Interlocking Factorization) for the solution of the symmetric linear systems is given by Evans [2]. Existence and stability of this factorization are proved by Khazal [3], when A is symmetric positive definite. Similar to the factorization given in [2], we present a matrix factorization $A = WW^{\mathrm{T}}$ for the solution of symmetric positive definite linear systems. Note that here, the structure of W appears as the transpose of the structure of W in [2]. The computation of the elements of W proceeds from middle outwards unlike outward middle as in [2]. The backward error analysis of the present method is given. The factorization $A = WW^{\mathrm{T}}$ involves n square root evaluations. A square root free WW^{T} factorization called WDW^{T} factorization is also presented.

The outline of the paper is as follows. In Section 2, we present the method. Algorithm and its complexity is given in Section 3. Existence results are included in Section 4 and backward error analysis is presented in Section 5. In Section 6 square root free WW^{T} factorization is presented. A numerical example is given in Section 7.

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2. Present method

Consider the linear system (1.1), where A is an $n \times n$ symmetric positive definite matrix with elements $A = (a_{i,j}), i,j = 1,2,\dots,n$, and x,f are n-component unknown and known vectors given by
$x = [x_1,\dots,x_n]^{\mathrm{T}}, f = [f_1,\dots,f_n]^{\mathrm{T}}$. Suppose $n = 2m-2$.

Assume that there exists a matrix W such that

$$A = WW^{\mathrm{T}}, \tag{2.1}$$

where

$$W = \begin{bmatrix} w_{1,1} & w_{1,2} & & \cdots & \cdots & & w_{1,n} \\ 0 & w_{2,2} & & \cdots & \cdots & w_{2,n-1} & 0 \\ & 0 & \ddots & & & \iddots & 0 & \\ & & & w_{m-1,m-1} & w_{m-1,m} & 0 & & \\ \vdots & \vdots & 0 & 0 & w_{m,m} & 0 & \vdots & \vdots \\ & & \iddots & & & \ddots & 0 & \\ & 0 & w_{n-1,3} & \cdots & \cdots & w_{n-1,n-1} & 0 \\ 0 & w_{n,2} & & \cdots & \cdots & & w_{n,n} \end{bmatrix}.$$

Note that the structure of W here, appears as the transpose of the W structure of [2].

Suppose $w_1,\dots,w_n$ are columns of W, then we have $W = [w_1,\dots,w_n]$. Each w_i, $i = 1,2,\dots,n$ is of the following form.

$$w_i = \begin{cases} [w_{1,i},\dots,w_{i,i},0,\dots,0,w_{n-i+2,i},\dots,w_{n,i}]^{\mathrm{T}} & \text{for } i = 1 \text{ to } m-1. \\ [w_{1,i},\dots,w_{n-i+1,i},0,\dots,0,w_{i,i},\dots,w_{n,i}]^{\mathrm{T}} & \text{for } i = m \text{ to } n. \end{cases}$$

The solution of the linear system $Ax = f$, is obtained by solving the two alternate subsystems

$$Wy = f \tag{2.2}$$

and

$$W^{\mathrm{T}}x = y. \tag{2.3}$$

The elements of W are calculated from middle to outward. The solution process with the coefficient matrix W proceeds from the middle towards the first and last unknowns, and the solution process with the coefficient matrix W^{T} proceeds from the first and last unknowns towards middle.

3. Algorithm and its complexity

The algorithm which executes the present method, is as follows.

Step 1. Factorization stage
for $k = 1$ to $m-1$
$w_{m+k-1,m+k-1} = \sqrt{a^{(k)}_{m+k-1,m+k-1}}$
$w_{i,m+k-1} = a^{(k)}_{i,m+k-1}/w_{m+k-1,m+k-1}$; $i = 1$ to $m-k$ and $m+k$ to $2m-2$
$w_{m-k,m-k} = \sqrt{a^{(k)}_{m-k,m-k} - w^2_{m-k,m+k-1}}$
$w_{i,m-k} = (a^{(k)}_{i,m-k} - w_{i,m+k-1}w_{m-k,m+k-1})/w_{m-k,m-k}$; $i = 1$ to $m-k-1$ and $m+k$ to $2m-2$
if $(k \neq m-1)$
$A_{k+1} = A_k - w_{m+k-1}w^{\mathrm{T}}_{m+k-1} - w_{m-k}w^{\mathrm{T}}_{m-k}$
end if
end for
Step 2. Solution of the subsystem $Wy = f$
for $m \leqslant i \leqslant n$
$y_i = (f_i - \sum_{k=n-i+2}^{i-1} w_{i,k}y_k)/w_{i,i}$
$y_{n+1-i} = (f_{n+1-i} - \sum_{k=n-i+2}^{i} w_{n+1-i,k}y_k)/w_{n+1-i,n+1-i}$
end for
Step 3. Solution of the subsystem $W^{\mathrm{T}}x = y$
for $1 \leqslant i \leqslant m-1$
$x_i = (y_i - \sum_{k=1}^{i-1} w_{k,i}x_k - \sum_{k=n+2-i}^{n} w_{k,i}x_k)/w_{i,i}$
$x_{n+1-i} = (y_{n+1-i} - \sum_{k=1}^{i} w_{k,n+1-i}x_k - \sum_{k=n+2-i}^{n} w_{k,n+1-i}x_k)/w_{n+1-i,n+1-i}$
end for