Applied Mathematics and Computation 203 (2008) 387–395

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Homotopy analysis method for the sine-Gordon equation with initial conditions

Uğur Yücel

Department of Mathematics, Faculty of Science and Art, Pamukkale University, Denizli 20020, Turkey

ARTICLE INFO

Keywords:
Homotopy analysis method
Sine-Gordon equation
Variational iteration method
Homotopy perturbation method

ABSTRACT

In this work, approximate analytical solution of the sine-Gordon equation with initial conditions is obtained by the homotopy analysis method (HAM). The HAM solutions contain an auxiliary parameter which provides a convenient way of controlling the convergence region of the series solutions. It is shown that the solutions obtained by the variational iteration method (VIM) are only special cases of the HAM solutions.

1. Introduction

The sine-Gordon equation, which is one of the basic nonlinear evolution equations,

$$u_{tt} - c^2 u_{xx} + \kappa \sin(u) = 0, \quad x \in R, \ t > 0, \tag{1}$$

where c and κ are constants, has arisen classically in the study of differential geometry, and in the propagation of a 'slip' dislocation in crystals [1]. More recently, as summarized in [1], it arises in a wide variety of physical problems including the propagation of magnetic flux in Josepson-type super conducting tunnel junctions, the phase jump of the wave function of superconducting electrons along long Josepson junctions [2,3], a chain of rigid pendula connected by springs [3], propagation of short optical pulses in resonant laser media [4,5], stability of fluid motions [6,7], in ferromagnetism and ferroelectric materials, in the dynamics of certain molecular chains such as DNA [8], in elementary particle physics [9–11], and in weakly unstable baroclinic wave packets in a two-layer fluid [12].

Various methods for finding particular solutions of the sine-Gordon equation (1) were developed in the latter part of the nineteenth century. One of the methods is known as Bäcklund transformations. Other methods include traveling wave solutions, the similarity method, the inverse scattering method, and the method of separation of variables, which deal with the representation of solutions as functions of independent variables.

There has recently been much attention devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytical or numerical, to the sine-Gordon equation. Herbst and Ablowitz [13] presented the numerical results of the sine-Gordon equation obtained by means of an explicit symplectic method. Ablowitz et al. [14] investigated the numerical behavior of a double-discrete, completely integrable discretization of the sine-Gordon equation and they illustrated their technique by numerical experiments. Wazwaz [15] used the tanh method to obtain the exact solutions of the sine-Gordon and the sinh-Gordon equations. Kaya [16] used the modified decomposition method and Batiha et al. [17] applied variational iteration method (VIM) to find the approximate analytical solution of the sine-Gordon equation.

E-mail address: uyucel@pau.edu.tr

0096-3003/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2008.04.042

Another powerful analytical method, called the homotopy analysis method (HAM), is a promising method for linear and nonlinear problems in science and engineering. It was first proposed by Liao [18]. The HAM contains an auxiliary parameter $\hbar$ which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called $\hbar$ curve, it is easy to find the valid regions of $\hbar$ to gain a convergent series solution. Thus, through HAM explicit analytic solutions of nonlinear problems are possible. A systematic and clear exposition on HAM is given in [19]. In recent years, this method has been successfully employed to solve many types of linear and nonlinear problems in science and engineering. Some of these works can be found in [20–27].

In this work, we present a reliable algorithm based on HAM to obtain approximate analytical solutions of the sine-Gordon equation with the following initial conditions

$$\begin{aligned} &u(x,0) = f(x), \\ &u_t(x,0) = g(x), \end{aligned} \tag{2}$$

where $f(x)$ and $g(x)$ are known functions.

We shall demonstrate that the solutions obtained by the VIM [17] is only special cases of the HAM solutions presented in this paper.

2. Homotopy analysis method (HAM)

To describe the basic ideas of the HAM, we consider the following differential equation:

$$N[u(x,t)] = 0, \tag{3}$$

where N is a nonlinear operator, x and t denote independent variables, and $u(x,t)$ is an unknown function. By means of generalizing the traditional homotopy method, Liao [19] constructs the so-called *zero-order deformation equation*

$$(1-p)L[\Phi(x,t;p) - u_0(x,t)] = p\hbar N[\Phi(x,t;p)], \tag{4}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary parameter, L is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\Phi(x,t;p)$ is an unknown function. It is important to note that one has great freedom to choose auxiliary objects such as $\hbar$ and L in HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\Phi(x,t;0) = u_0(x,t), \quad \Phi(x,t;1) = u(x,t), \tag{5}$$

respectively. Thus, as the embedding parameter p increases from 0 to 1, the solution $\Phi(x,t;p)$ varies (or deforms) continuously from the initial guess $u_0(x,t)$ to the exact solution $u(x,t)$ of the original equation (3).

Expanding $\Phi(x,t;p)$ in Taylor series with respect to p, one has

$$\Phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)p^m, \tag{6}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \Phi(x,t;p)}{\partial p^m}\bigg|_{p=0}. \tag{7}$$

If the auxiliary linear operator, the initial guess, and the auxiliary parameter $\hbar$ are so properly chosen, then, as proved by Liao [19], the series (6) converges at $p = 1$ and one has

$$\Phi(x,t;1) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t). \tag{8}$$

Therefore, using Eq. (5), we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t). \tag{9}$$

This expression provides us with a relationship between the initial guess $u_0(x,t)$ and the exact solution $u(x,t)$ by means of the terms $u_m(x,t)$ $(m = 1,2,3,\ldots)$ which are determined by the so-called high-order deformation equations described below.

For brevity, define the vector

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_n(x,t)\}. \tag{10}$$

According to definition (7), the governing equation of $u_m(x,t)$ can be derived from the zero-order deformation equation (4). Differentiating the zero-order deformation equation (4) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called *mth-order deformation equation,*

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m(\vec{u}_{m-1}), \tag{11}$$