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N-soliton solutions for the combined KdV–CDG equation and the KdV–Lax equation

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ABSTRACT

The Hirota-Ito approach presented in [R. Hirota, M. Ito, Resonance of solitons in one dimension, J. Phys. Soc. Jpn. 52(3) (1983) 744–748] for extending fifth-order integrable equations with a nonvanishing boundary conditions to combined equations is used in this work. The generalized fifth-order Caudrey-Dodd-Gibbon (CDG) and Lax equations are extended to combined integrable equations. The Hirota's bilinear method is used to derive multiple-soliton solutions for the extended KdV–CDG and KdV–Lax equations.

1. Introduction

Hirota and Ito [1] investigated the resonance of solitons in one-dimensional space. Three significant conclusions were made in this work concerning two solitons at the resonant case, near the resonant case, and becoming singular solitons where sech^2 profiles are transmitted with cosech^2 profiles through interaction.

The generalized fifth-order KdV (fKdV) equation reads

$$u_t + au^2u_x + au_xu_{xx} + cuu_{3x} + u_{5x} = 0, \tag{1}$$

where a, b, and c are arbitrary nonzero and real parameters, and $u = u(x,t)$ is a sufficiently smooth function. The fifth-order KdV equation includes two dispersive terms u_{3x} and u_{5x}. A variety of the fKdV equations can be developed by changing the real values of the parameters a, b, and c. The most well-known fifth-order KdV equations that will be approached are the Sawada-Kotera (SK) equation, the Caudrey-Dodd-Gibbon equation [2], the Lax equation [3], the Kaup-Kupershmidt (KP) equation, and the Ito equation. The derivation of these fifth-order forms are derived from specific bilinear forms of the so-called Hirota's D-operators. In this work, we will conduct our study on two of these equations, the Caudrey-Dodd-Gibbon equation (CDG) [2], and the Lax equation [3].

2. Caudrey-Dodd-Gibbon equation

The Caudrey-Dodd-Gibbon equation (CDG) [2] is given by

$$u_t + \frac{1}{5}\alpha^2u^2u_x + \alpha u_xu_{2x} + \alpha uu_{3x} + u_{5x} = 0, \tag{2}$$

with $u(x,t)$ is a sufficiently often differentiable function. The CDG equation is completely integrable and therefore it admits multiple-soliton solutions and infinite number of conserved quantities. Moreover, the CDG Eq. (2) possesses the Painlevé property. Following [1], we consider the CDG equation with a nonvanishing boundary condition

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$$u|_{x\simeq\pm\infty} = c, \tag{3}$$

where c is a constant.

The CDG Eq. (2) can be written as

$$u_t + b\left(\frac{1}{15}\alpha^2 u^3 + \alpha u u_{2x} + u_{4x}\right)_x = 0, \tag{4}$$

where b is a constant. Following the approach in [1], we replace u by $u + c$ that will carry out (4) to

$$u_t + b\left(\frac{1}{15}\alpha^2 (u+c)^3 + \alpha(u+c) u_{2x} + u_{4x}\right)_x = 0, \tag{5}$$

so that

$$u_t + bc\alpha\left[u_{2x} + \frac{1}{5}\alpha u^2 + \frac{1}{5}\alpha c u\right]_x + b\left[\frac{1}{15}\alpha^2 u^3 + \alpha u u_{2x} + u_{4x}\right]_x = 0. \tag{6}$$

We next set

$$c = \frac{a}{b\alpha}, \tag{7}$$

and by using the Galilei transformation, to get rid of u_x, Eq. (6) becomes

$$u_t + a\left[u_{2x} + \frac{1}{5}\alpha u^2\right]_x + b\left[\frac{1}{15}\alpha^2 u^3 + \alpha u u_{2x} + u_{4x}\right]_x = 0. \tag{8}$$

Eq. (8) will be reduced to the KdV equation [4] for $b = 0$ and for the CDG Eq. (5) for $a = 0$. Moreover, Eq. (8) is completely integrable. It is our aim in this work to show that this equation exhibits N-soliton solutions. Eq. (8) has the bilinear form

$$D_x(D_t + aD_x^3 + bD_x^5)f \cdot f = 0, \tag{9}$$

where the customary definition of the Hirota's bilinear operators D is given by

$$D_t^n D_x^m a.b = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m a(x,t)b(x',t')|x' = x, t' = t. \tag{10}$$

The solution $u(x,t)$ is defined by

$$u(x,t) = R(\ln f)_{xx}, \tag{11}$$

where R is a parameter that will be determined. The auxiliary $f(x,t)$ is given by the perturbation expansion

$$f(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n(x,t), \tag{12}$$

where ϵ is a bookkeeping non-small parameter, and $f_n(x,t), n = 1, 2, \ldots$ are unknown functions that will be determined by substituting the last equation into the bilinear form and solving the resulting equations by equating different powers of ϵ to zero.

In what follows we briefly highlight the main features of the Hirota's bilinear method that will be used in this work [4–20]. We first substitute

$$u(x,t) = e^{kx - ct}, \tag{13}$$

into the linear terms of any equation under discussion to determine the dispersion relation between k and c. We then substitute the single soliton solution

$$u(x,t) = R(\ln f)_{xx} = R\frac{ff_{2x} - (f_x)^2}{f^2}, \tag{14}$$

into the equation under discussion, where the auxiliary function f is given by

$$f(x,t) = 1 + f_1(x,t) = 1 + e^{\theta_1}, \tag{15}$$

where

$$\theta_i = k_i x - c_i t, \quad i = 1, 2, \ldots, N, \tag{16}$$

and by solving the resulting equation, we determine the numerical value for R [21–30]. The steps are summarized by

(i) For dispersion relation, we use

$$u(x,t) = e^{\theta_i}, \quad \theta_i = k_i x - c_i t. \tag{17}$$