Applied Mathematics and Computation 215 (2009) 916–927

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Shape controlled interpolatory ternary subdivision

C. Beccari [a,*], G. Casciola [a], L. Romani [b]

[a] Department of Mathematics, University of Bologna, P.zza di Porta San Donato 5, 40127 Bologna, Italy
[b] Department of Mathematics and Applications, University of Milano-Bicocca, Via R. Cozzi 53, 20125 Milano, Italy

A R T I C L E I N F O

Keywords:
Univariate ternary refinement
Interpolation
Non-stationarity
Tension control
Conic section
Combined subdivision scheme

A B S T R A C T

Ternary subdivision schemes compare favorably with their binary analogues because they are able to generate limit functions with the same (or higher) smoothness but smaller support.

In this work we consider the two issues of local tension control and conics reproduction in univariate interpolating ternary refinements. We show that both these features can be included in a unique interpolating 4-point subdivision method by means of non-stationary insertion rules that do not affect the improved smoothness and locality of ternary schemes. This is realized by exploiting local shape parameters associated with the initial polyline edges.

1. Introduction

The interest in investigating arities higher than two has started from the seminal paper by Hassan et al. [1]. Here they showed that, for the so-called interpolating 4-point stationary scheme, higher smoothness and smaller support can be achieved by going from binary to ternary.

At each iteration $k \geqslant 0$, an interpolating 4-point ternary subdivision scheme maps the polygon $P^k = \{p_j^k\}_{j\in\mathbb{Z}}$ to the refined polygon $P^{k+1} = \{p_j^{k+1}\}_{j\in\mathbb{Z}}$ through the insertion rules

$$\begin{aligned} p_{3j}^{k+1} &= p_j^k, \\ p_{3j+1}^{k+1} &= a_0^k p_{j-1}^k + a_1^k p_j^k + a_2^k p_{j+1}^k + a_3^k p_{j+2}^k, \\ p_{3j+2}^{k+1} &= a_3^k p_{j-1}^k + a_2^k p_j^k + a_1^k p_{j+1}^k + a_0^k p_{j+2}^k, \end{aligned} \tag{1}$$

where the coefficients $\{a_i^k\}_{i=0,1,2,3}$ are chosen in order to satisfy the relation

$$a_0^k + a_1^k + a_2^k + a_3^k = 1. \tag{2}$$

In general, the coefficients may either stay constant throughout the subdivision process or change according to the refinement level k. The scheme is said *stationary* in the first case and *non-stationary* in the latter.

Although Hassan's stationary refinement still looks like the most appealing proposal of interpolatory ternary subdivision, it does not provide great design flexibility. In fact, once the initial control points are chosen, this scheme does not allow the user to control the shape of the interpolant. Moreover, although such a scheme reproduces cubic polynomials starting from uniformly spaced initial samples, it does not reproduce any other analytic curve.

* Corresponding author.
E-mail addresses: beccari@dm.unibo.it (C. Beccari), casciola@dm.unibo.it (G. Casciola), lucia.romani@unimib.it (L. Romani).

0096-3003/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2009.06.014

The goal of this paper is to improve the design performance of the 4-point ternary interpolating scheme, without affecting its improved smoothness and local support. We will thus consider two kinds of shape manipulations: one for controlling the tension of the limit curve in correspondence to each edge of the initial polygon and another for exactly reproducing circles and more generally conic sections. The first has a two-fold advantage: it allows us to arbitrarily modify the shape of the interpolant and, if the initial tension values are set properly, it may alleviate the undesired undulations that often arise as a consequence of the interpolation process. The second one, i.e. the capability of reproducing conic sections, is obviously a fundamental feature in many application contexts.

Our aim is defining an interpolating algorithm that reproduces conic sections in those regions of the initial polyline where the given samples belong to one of these curves and locally-controlled C^2 curves otherwise.

The definition of a locally-controlled interpolating ternary subdivision scheme has been already addressed in [2]. In that work, a *tension parameter* is associated with each edge of the initial polyline. In this way, by progressively increasing/decreasing its value, in the corresponding region of interest the limit curve tends to become tighter/looser to the underlying data polygon. After one initial parameter has been assigned to each edge, its value is automatically updated at each iteration by means of a recurrence formula. The corresponding refinement algorithm is non-stationary and generates C^2-continuous limit curves for any choice of the initial tensions in a wide span of definition. However, it does not reproduce the whole class of conic sections nor any other analytic curve except quadratic polynomials.

In binary subdivision literature, the construction of subdivision schemes reproducing conic sections has been addressed extensively [3–10]. Conversely, in the ternary context, there is still no available scheme for exactly generating these families of curves. The only attempt towards the definition of a ternary circle-preserving subdivision scheme has appeared in [11]. In that work, exploiting a refinement which preserves the discrete curvature and tangent direction at each vertex of the starting control polygon, a satisfactory approximation of a circle is generated starting from a regular κ-gon. However, the derived algorithm is quite involved, computationally heavy and, as acknowledged by the authors themselves, the smoothness of the related limit curves is also not known.

Thus, to the aim of exactly generating conic sections, we first address the construction of a non-stationary interpolating 4-point ternary refinement where the insertion rules are determined through Lagrange interpolation by functions from the 4-dimensional space $\{1, x, e^{tx}, e^{-tx}\}$, where $t \in \{0, s, is | s > 0\}$. As a consequence of its definition, such a scheme reproduces certain exponential functions and in particular circles and conic sections (whenever the initial points are sampled at equally-spaced values of t on these kinds of curves and the starting parameters are initialized accordingly). Similarly to the ternary 4-point scheme presented in [2], this refinement is non-stationary and its level-dependent coefficients are automatically computed by means of a recurrence formula based on a third-angle relation linking parameter values at two successive subdivision steps. We show that the proposed scheme generates C^1-continuous limit curves and we discuss how its property of reproducing conic sections depends on an opportune choice of the initial parameter.

We finally present the unified subdivision scheme that combines the locally-controlled C^2 scheme in [2] with the derived exponentials reproducing scheme. The proposed algorithm is realized by allowing different regions of the initial polyline to be refined with one subdivision scheme or the other. Provided that the initial parameters are set properly, the combined scheme defined in this way generates arcs of conic sections where needed and locally-controlled C^2 curves otherwise. Moreover, in the region common to both schemes, its limit curves are C^1-continuous.

The paper is organized as follows: In Section 2, we start out by reviewing the locally-controlled C^2 subdivision scheme presented in [2]. In Section 3, we define the novel ternary scheme reproducing exponentials (Section 3.1), analyze its smoothness (Section 3.2) and discuss how its parameter needs to be initialized according to the spacing of the given samples (Section 3.3). In Section 4, we finally present the refinement algorithm that unifies the two schemes described in Sections 2 and 3. Then we analyze its smoothness and illustrate some application examples.

2. A locally-controlled C^2 interpolating 4-point ternary subdivision scheme

The scheme in [2] is summarized by the following procedure. At the first stage, the polyline $P^0 = \{p_j^0\}_{j\in\mathbb{Z}}$ and an initial set of parameters $\{\upsilon_j^0\}_{j\in\mathbb{Z}}$ are given. The value υ_j^0 is assigned to the edge $\overline{p_j^0 p_{j+1}^0}$.

At the iteration $k \geqslant 0$, the coarse polyline P^k is transformed into the refined polyline P^{k+1} by applying the 4-point insertion rules

$$\begin{aligned} p_{3j}^{k+1} &= p_j^k, \\ p_{3j+1}^{k+1} &= a_{0,j}^k p_{j-1}^k + a_{1,j}^k p_j^k + a_{2,j}^k p_{j+1}^k + a_{3,j}^k p_{j+2}^k, \\ p_{3j+2}^{k+1} &= a_{3,j}^k p_{j-1}^k + a_{2,j}^k p_j^k + a_{1,j}^k p_{j+1}^k + a_{0,j}^k p_{j+2}^k, \end{aligned} \tag{3}$$

with coefficients

$$\begin{aligned} a_{0,j}^k &= \frac{1}{60}(-90u_j^{k+1} - 1) \quad a_{1,j}^k = \frac{1}{60}(90u_j^{k+1} + 43), \\ a_{2,j}^k &= \frac{1}{60}(90u_j^{k+1} + 17) \quad a_{3,j}^k = \frac{1}{60}(-90u_j^{k+1} + 1), \end{aligned} \tag{4}$$