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A Matlab-based rapid method for computing lattice-subspaces and vector sublattices of $\mathbb{R}^n$: Applications in portfolio insurance

Vasilios N. Katsikis *

Department of Mathematics, National Technical University of Athens, Zographou 15780, Athens, Greece
Department of Applied Sciences, TEI of Chalkis, GR 34400 Psahna, Greece

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ABSTRACT

This paper provides the construction of a powerful and efficient computational method, that translates Polyrakis algorithm [I.A. Polyrakis, Minimal lattice-subspaces, Trans. Am. Math. Soc. 351 (1999) 4183–4203, Theorem 3.19] for the calculation of lattice-subspaces and vector sublattices in $\mathbb{R}^n$. In the theory of finance, lattice-subspaces have been extensively used in order to provide a characterization of market structures in which the cost-minimizing portfolio is price-independent. Specifically, we apply our computational method in order to solve a cost minimization problem that ensures the minimum-cost insured portfolio.

1. Introduction

In [11,12], lattice-subspaces and sublattices of the space of continuous real valued functions $C(\Omega)$ defined on a compact Hausdorff topological space Ω are studied. In the case where Ω is finite, for example $\Omega = \{1, 2, \ldots, n\}$, then $C(\Omega) = \mathbb{R}^n$ and the results of [11,12] can be applied for the determination of the lattice-subspaces and sublattices of $\mathbb{R}^n$. In the present manuscript we implement, in Matlab, these methods and we apply these results in finance. In particular, we present a new and more efficient computational method, than the one presented in [9], in order to check whether a finite collection of positive, linearly independent vectors of $\mathbb{R}^n$ generates a lattice-subspace or a vector sublattice, within Matlab, by introducing a new function named `SUBlatSUB`. Matlab is an established tool for scientists that provides ready access to many mathematical models in different scientific areas. Our main concern has been to make lattice-subspaces and vector sublattices of $\mathbb{R}^n$ as easily accessible to the interested user especially in the field of applications such as portfolio insurance. In a previous article [9], we provided a computational method in order to solve the same problem in $\mathbb{R}^n$ by following an algorithm that was proposed in [1]. The Matlab function (see the **K** function in [9]) for this algorithm was an elegant, fast and accurate tool in order to provide whether or not the given collection of vectors forms a lattice-subspace. The new function `SUBlatSUB` features explicit solutions to the problem while the computational effort required in order to obtain the lattice-subspace or the vector sublattice is substantially lower, particularly for large collections of vectors. In addition, besides the effectiveness of this new method, we are also able to check whether the given collection forms a vector sublattice. We also show that `SUBlatSUB` is much faster when the given collection of vectors defines a vector sublattice rather than a lattice-subspace since, in fact, the joint amount of tests, calculations and further considerations required to reach the goal is substantially less.

As in [9], we intend to apply our new computational method in Mathematical Economics, especially in the areas of incomplete markets and portfolio insurance as they are described in [2–4]. So, Section 3 discusses an investment strategy called

* Address: Department of Mathematics, National Technical University of Athens, Zographou 15780, Athens, Greece.
E-mail addresses: vaskats@gmail.com, vaskats@mail.ntua.gr

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minimum-cost portfolio insurance as a solution of a cost minimization problem and how our new proposed method translates the economics problem into the language of computing.

For calculations both on the lattice-subspace or the vector sublattice generated by a finite collection of positive, linearly independent vectors of $\mathbb{R}^n$, as well as on the minimum-cost insured portfolio problem the high-end Matlab programming language was used. Specifically, the Matlab 7.4 (R2007b) environment [7,8] was used on an Intel(R) Pentium(R) Dual CPU T2310 @ 1.46 GHz 1.47 GHz 32-bit system with 2 GB of RAM memory running on the Windows Vista Home Premium Operating System.

2. Lattice-subspaces and vector sublattices of R^n

2.1. Preliminaries

We first recall some definitions and notation from the vector lattice theory. Let $\mathbb{R}^n = \{x = (x(1), x(2), \ldots, x(n)) | x(i) \in \mathbb{R},\ \text{for each } i\}$, where we view $\mathbb{R}^n$ as an ordered space. Then, the *pointwise order* relation in $\mathbb{R}^n$ is defined by

$$x \leqslant y \quad \text{if and only if } x(i) \leqslant y(i), \quad \text{for each } i = 1, \ldots, n.$$

The positive cone of $\mathbb{R}^n$ is defined by $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x(i) \geqslant 0,\ \text{for each } i\}$ and if we suppose that X is a vector subspace of $\mathbb{R}^n$ then X ordered by the pointwise ordering is an *ordered subspace* of $\mathbb{R}^n$ with positive cone X_+ defined by $X_+ = X \cap \mathbb{R}^n_+$. A point $x \in \mathbb{R}^n$ is an *upper bound (lower bound)* of a subset $S \subseteq \mathbb{R}^n$ if and only if $y \leqslant x (x \leqslant y)$, for all $y \in S$. For a two-point set $S = \{x, y\}$, we denote by $x \vee y (x \wedge y)$ the *supremum* of S i.e., its least upper bound(the *infimum* of S i.e., its greatest lower bound). Thus, $x \vee y (x \wedge y)$ is the componentwise maximum(minimum) of x and y defined by

$$(x \vee y)(i) = \max\{x(i), y(i)\}((x \wedge y)(i) = \min\{x(i), y(i)\}), \quad \text{for all } i = 1, \ldots, n.$$

An ordered subspace X of $\mathbb{R}^n$ is a *lattice-subspace* of $\mathbb{R}^n$ if it is a vector lattice in the induced ordering, i.e., for any two vectors $x, y \in X$ the supremum and the infimum of $\{x, y\}$ both exist in X. Note that the supremum and the infimum of the set $\{x, y\}$ are, in general, different in the subspace than the supremum and the infimum of this set in the initial space. An ordered subspace Z of $\mathbb{R}^n$ is a *vector sublattice* or a *Riesz subspace* of $\mathbb{R}^n$ if for any $x, y \in Z$ the supremum and the infimum of the set $\{x, y\}$ in $\mathbb{R}^n$ belong to Z.

Suppose that X is an ordered subspace of $\mathbb{R}^n$ and $B = \{b_1, b_2, \ldots, b_m\}$ is a basis for X. Then B is a *positive basis*of X if for each $x \in X$ it holds that x is positive if and only if its coefficients in the basis B are positive. In other words, B is a positive basis of X if the positive cone X_+ of X has the form,

$$X_+ = \{x = \sum_{i=1}^{m} \lambda_i b_i | \lambda_i \geqslant 0,\ \text{for each } i\}.$$

Therefore, if $x = \sum_{i=1}^m \lambda_i b_i$ and $y = \sum_{i=1}^m \mu_i b_i$ then $x \leqslant y$ if and only if $\lambda_i \leqslant \mu_i$ for each $i = 1, 2, \ldots, m$.

Each element b_i of the basis B is an extremal[1] point of X_+ thus a positive basis of X is unique in the sense of positive multiples. The existence of positive bases is not always ensured, but in the case where X is a vector sublattice of $\mathbb{R}^n$ then X has always a positive basis. Moreover, it holds that an ordered subspace of $\mathbb{R}^n$ has a positive basis if and only if it is a lattice-subspace of $\mathbb{R}^n$. If $B = \{b_1, b_2, \ldots, b_m\}$ is a positive basis for a lattice-subspace (or a vector sublattice) X then the lattice operations in X, namely $x \triangledown y$ for the supremum and $x \vartriangle y$ for the infimum of the set $\{x, y\}$ in X, are given by

$$x \triangledown y = \sum_{i=1}^{m} \max\{\lambda_i, \mu_i\} b_i \quad \text{and} \quad x \vartriangle y = \sum_{i=1}^{m} \min\{\lambda_i, \mu_i\} b_i$$

for each $x = \sum_{i=1}^m \lambda_i b_i, y = \sum_{i=1}^m \mu_i b_i \in X$. A vector sublattice is always a lattice-subspace, but the converse is not true as shown in the next example,

Example 1. Let $X = [x_1, x_2, x_3]$ be the subspace of $\mathbb{R}^4$ generated by the vectors $x_1 = (6, 0, 0, 1), x_2 = (6, 4, 0, 0), x_3 = (8, 4, 2, 0)$. An easy argument shows that the set $B = \{b_1, b_2, b_3\}$ where

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 12 & 8 & 0 & 0 \\ 6 & 0 & 0 & 1 \end{bmatrix}$$

forms a positive basis of X therefore X is a lattice-subspace of $\mathbb{R}^4$. On the other hand, let us consider the vectors $y_1 = 2x_1 + x_2 = (18, 4, 0, 2)$ and $y_2 = x_3 - x_2 = (2, 0, 2, 0)$ of X. Then, $y_1 \vee y_2 = (18, 4, 2, 2)$ and since $y_1 = \frac{1}{2} b_2 + 2b_3, y_2 = b_1$, it follows that $y_1 \triangledown y_2 = b_1 + \frac{1}{2} b_2 + 2b_3 = (20, 4, 4, 2)$, therefore, X is not a vector sublattice of $\mathbb{R}^4$.

For an extensive presentation of lattice-subspaces, vector sublattices and positive bases the reader may refer to [1,11,12].

[1] A nonzero element x_0 of X_+ is an *extremal point* of X_+ if, for any $x \in X, 0 \leqslant x \leqslant x_0$ implies $x = \lambda x_0$ for a real number λ.