Applied Mathematics and Computation 215 (2009) 1135–1140

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Summation formulae for finite cotangent sums

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ARTICLE INFO

Keywords:
Trigonometric sums
Finite summation
Cotangent sums
Alternate cotangent sums
Contour integration
Cauchy residue theorem
Higher order Bernoulli polynomials
Bernoulli polynomials
Bernoulli numbers

ABSTRACT

Recently, a half-dozen remarkably general families of the finite trigonometric sums were summed in closed-form by choosing a particularly convenient integration contour and making use of the calculus of residues. In this sequel, we show that this procedure can be further extended and we find the summation formulae, in terms of the higher order Bernoulli polynomials and the ordinary Bernoulli polynomials, for four general families of the finite cotangent sums.

1. Introduction

Recently, a half-dozen remarkably general families of the finite trigonometric sums were summed in closed-form by choosing a particularly convenient integration contour and making use of the calculus of residues [1–4].

In this sequel, we show that the following families of finite alternating cotangent sums

$$S^{*}_{2n+1}(q,r) := \sum_{p=1}^{q-1} (-1)^p \sin\left(\frac{2rp\pi}{q}\right) \cot^{2n+1}\left(\frac{p\pi}{q}\right) \tag{1.1}$$

($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$; *q is an even positive integer greater than* 2; $r = 1,\ldots,q-1$),

$$C^{*}_{2n}(q,r) := \sum_{p=1}^{q-1} (-1)^p \cos\left(\frac{2rp\pi}{q}\right) \cot^{2n}\left(\frac{p\pi}{q}\right) \tag{1.2}$$

($n \in \mathbb{N}$; *q is an even positive integer greater than* 2; $r = 1,\ldots,q-1$)

and

$$C^{*}_{2n}(q) := \sum_{p=1}^{q-1} (-1)^p \cot^{2n}\left(\frac{p\pi}{q}\right) \tag{1.3}$$

($n \in \mathbb{N}$; *q is an even positive integer greater than* 2),
as well as the family of sums given in (2.14) below, can be considered in the same way and we find their closed-form summation formulae in terms of the higher order Bernoulli polynomials and the ordinary Bernoulli polynomials and numbers.

2. Statement of main results

Observe that, throughout the text, we set an empty sum to be zero. We use the floor function $\lfloor x \rfloor$, also called the greatest integer function or integer value, which gives the largest integer less than or equal to x.

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doi:10.1016/j.amc.2009.06.053

In what follows, we denote by $B_n^{(m)}(x)$ the Bernoulli polynomial of order m and degree n defined by means of the following generating functions (see, for details, [5, p. 53, *et seq.*] and [6, Section 1.6])

$$\left(\frac{t}{e^t-1}\right)^m e^{tx} = \sum_{n=0}^{\infty} B_n^{(m)}(x)\frac{t^n}{n!}, \quad (|t|<2\pi; m \in \mathbb{N}_0). \tag{2.1}$$

For $m=1$ we have

$$B_n(x) := B_n^{(1)}(x), \quad (n \in \mathbb{N}_0), \tag{2.2}$$

where $B_n(x)$ is the relatively more familiar (ordinary) Bernoulli polynomial (see, for instance, [5, p. 35, *et seq.*]). The (ordinary) Bernoulli number B_n is given by

$$B_n := B_n(0) \quad (n \in \mathbb{N}_0). \tag{2.3}$$

Our results are as follows.

Theorem 1. *Let $B_n^{(m)}(x)$ be the Bernoulli polynomial of order m and degree n defined by (2.1) and let $B_n(x)$ be the Bernoulli polynomial defined as in (2.2). Let $\delta_{i,j}$ be the Kronecker delta and suppose that $\{x\} = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the greatest integer function.*

Then, the sums $S_{2n+1}^*(q,r)$ in (1.1) are given by

$$S_{2n+1}^*(q,r) = \mathscr{S}_{2n+1}^*(q,r), \tag{2.4}$$

where

$$\mathscr{S}_{2n+1}^*(q,r) = (-1)^{n-1}q\delta_{q,2r} + \frac{(-1)^{n-1}}{(2n+1)!}\cdot\sum_{\alpha=0}^{n}\sum_{\beta=0}^{2n+1}\binom{2n+1}{2\alpha+1}\binom{2n+1}{\beta}B_{2\alpha+1}\left(\left\{\frac{r}{q}+\frac{1}{2}\right\}\right)B_{2n-2\alpha}^{(2n+1)}(\beta)q^{2\alpha+1} \tag{2.5}$$

($n \in \mathbb{N}_0$; q is an even positive integer greater than 2; $r = 1,\ldots,q-1$),
while the sums $C_{2n}^*(q,r)$ in (1.2) are given by

$$C_{2n}^*(q,r) = \mathscr{C}_{2n}^*(q,r), \tag{2.6}$$

where

$$\mathscr{C}_{2n}^*(q,r) = (-1)^{n}q\delta_{q,2r} + \frac{(-1)^{n-1}}{(2n)!}\sum_{\alpha=0}^{n}\sum_{\beta=0}^{2n}\binom{2n}{2\alpha}\binom{2n}{\beta}B_{2\alpha}\left(\left\{\frac{r}{q}+\frac{1}{2}\right\}\right)B_{2n-2\alpha}^{(2n)}(\beta)q^{2\alpha} \tag{2.7}$$

($n \in \mathbb{N}$; q is an even positive integer greater than 2; $r = 1,\ldots,q-1$).

Theorem 2. *Let $B_n^{(m)}(x)$ be the Bernoulli polynomial of order m and degree n defined by (2.1) and let $B_n(x)$ be the Bernoulli polynomial defined as in (2.2). Then, the sums*

$$S_{2n+1}(q,r) := \sum_{p=1}^{q-1}\sin\left(\frac{2rp\pi}{q}\right)\cot^{2n+1}\left(\frac{p\pi}{q}\right) \tag{2.8}$$

$$(n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\};\ q \in \mathbb{N}\setminus\{1\};\ r = 1,\ldots,q-1)$$

are given by

$$S_{2n+1}(q,r) = \mathscr{S}_{2n+1}(q,r), \tag{2.9}$$

where

$$\mathscr{S}_{2n+1}(q,r) = \frac{(-1)^{n-1}}{(2n+1)!}\cdot\sum_{\alpha=0}^{n}\sum_{\beta=0}^{2n+1}\binom{2n+1}{2\alpha+1}\binom{2n+1}{\beta}B_{2\alpha+1}\left(\frac{r}{q}\right)B_{2n-2\alpha}^{(2n+1)}(\beta)q^{2\alpha+1} \tag{2.10}$$

($n \in \mathbb{N}_0$; $q \in \mathbb{N}\setminus\{1\}$; $r = 1,\ldots,q-1$),

while the sums

$$C_{2n}(q,r) := \sum_{p=1}^{q-1}\cos\left(\frac{2rp\pi}{q}\right)\cot^{2n}\left(\frac{p\pi}{q}\right) \quad (n \in \mathbb{N};\ q \in \mathbb{N}\setminus\{1\};\ r = 1,\ldots,q-1) \tag{2.11}$$

are given by

$$C_{2n}(q,r) = \mathscr{C}_{2n}(q,r), \tag{2.12}$$

where