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The estimate for number of zeros of solutions of second order functional differential equations

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ABSTRACT

In this paper, by extending the maximum principle, we study the number of zeros of solutions of second order functional differential equations. We obtain a sufficient condition for the existence of at most one zero of solutions on an interval. On this basis, we estimate the maximal number of zeros of solutions on a large interval. For illustrating the theoretical analysis, we also give two numerical simulation examples.

1. Introduction

Oscillatory theory of functional differential equation has been extensively developed during past few decades. We refer to the reader to the monographs [1–4]. Detailed results about estimates for number of zeros of solutions in an interval have also been obtained in the literature. Particularly, in [5,6], the authors investigated the number of zeros of the ordinary differential equations

$$x'' + p(t)x = 0 \tag{1}$$

and

$$(r(t)x')' + p(t)x = 0. \tag{2}$$

In [7], the authors considered the second order ordinary differential equation

$$x'' + g(t)x' + h(t)x = 0 \tag{3}$$

and by using maximum principle gave some results for the solution of Eq. (3) having at most one zero on an interval. Various estimates of the number of zeros (or of distance between two adjacent zeros) of second order delay equation

$$x'' + \sum_{i=0}^{n} p_i(t)x(h_i(t)) = 0, \tag{4}$$

where $p_i \geqslant 0$, $h_i \leqslant t$ were obtained in [8–11]. Distributions of zeros for first order delay differential equation

$$x' + p(t)x(h(t)) = 0 \tag{5}$$

and neutral differential equation

$$(x + p(t)x(h(t)))' + q(t)x(r(t)) = 0, \tag{6}$$

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where $h(t) \leqslant t$, $r(t) \leqslant t$ were considered in the recent papers [12–19].

In this paper we consider the following equation

$$x'' + g(t, x, x_t)x' + h(t, x, x_t) = 0, \tag{7}$$

where $\mathbb{R} = (-\infty, \infty), [\alpha, \beta] \subset \mathbb{R}, t \in [\alpha, \beta]$. We suppose that $C([\alpha, \beta], \mathbb{R})$ is the space of continuous functions mapping the interval $[\alpha, \beta]$ into $\mathbb{R}$ with the norm $\|\varphi\| = \sup_{t\in[\alpha,\beta]}|\varphi(t)|$ for $\varphi \in C([\alpha, \beta], \mathbb{R})$ and that $C^k([\alpha, \beta], \mathbb{R})$ denotes a set of all functions $\varphi \in C([\alpha, \beta], \mathbb{R})$ with kth order continuous derivative on $[\alpha, \beta]$ where the derivatives at the end points α and β are the single-side ones. If $[\alpha, \beta] = [-r, 0]$ where r is a given real number and $r \geqslant 0$ we let $C = C([-r, 0], \mathbb{R})$. Suppose $x \in C^2([\alpha, \beta], \mathbb{R})$. For any $t \in [\alpha, \beta]$, we let x_t be defined by $x_t(\theta) = x(t + \theta), -r \leqslant \theta \leqslant 0$. If D is subset of $[\alpha, \beta] \times C([\alpha, \beta], \mathbb{R}) \times C$, we let $g, h : D \to \mathbb{R}$.

Our purpose is to study the number of zeros of solution of Eq. (7). By extending the maximum principle to the second order functional differential equation (7), we will obtain a sufficient condition for the existence of at most one zero of solution on an interval. For the illustration, we will give two numerical simulation examples and estimate the maximal number of zeros of their solutions on a given interval.

Throughout the paper, we suppose that $g(t, \phi, \varphi), h(t, \phi, \varphi)$ are continuous functions in t, ϕ and φ where $(t, \phi, \varphi) \in D$. We denote the Fréchet derivatives of $g(t, \phi, \varphi)$ and $h(t, \phi, \varphi)$ in ϕ and φ by $g'_\phi(t, \phi, \varphi), g'_\varphi(t, \phi, \varphi), h'_\phi(t, \phi, \varphi)$ and $h'_\varphi(t, \phi, \varphi)$ which are all continuous in t, ϕ and φ. Moreover, we say that $f(\cdot, \phi, \varphi)$ is C^1 local bounded on $[\alpha, \beta]$ if for any $[a, b] \subset (\alpha, \beta)$ and any $S(K_1) = \{\phi \in C([a, b], \mathbb{R}) | \|\phi\| \leqslant K_1\}$ and $S(K_2) = \{\varphi \in C | \|\varphi\| \leqslant K_2\}$ where $K_1, K_2 > 0$, there exist positive constants $N_{i,f}(K_1, K_2, a, b), i = 1, 2, 3$ such that

$$\|f(t, \phi, \varphi)\| \leqslant N_{1,f}(K_1, K_2, a, b),$$
$$\|f'_\phi(t, \phi, \varphi)\| \leqslant N_{2,f}(K_1, K_2, a, b)$$

and

$$\|f'_\varphi(t, \phi, \varphi)\| \leqslant N_{3,f}(K_1, K_2, a, b)$$

for $(t, \phi, \varphi) \in [a, b] \times S(K_1) \times S(K_2)$.

A function x is said to be a solution of Eq. (7) on $[\alpha - r, \beta]$ if there is $[\alpha, \beta] \subset \mathbb{R}$ such that $(t, x, x_t) \in [\alpha, \beta] \times C([\alpha - r, \beta], \mathbb{R}) \times C$ and $x(t)$ satisfies Eq. (7) for $t \in [\alpha, \beta]$.

Let $L_g[x] = x''(t) + g(t, x, x_t)x'(t)$ and $h[x] = h(t, x, x_t)$, then Eq. (7) may be the brief form as follows

$$(L_g + h)[x] = 0. \tag{8}$$

2. Main results

Lemma 2.1 [20, Lemma 2.1, p. 38]. *If $x \in C([\alpha - r, \beta], \mathbb{R})$, then x_t is a continuous function of t for $t \in [\alpha, \beta]$.*

Lemma 2.2. *Suppose $x \in C^2([\alpha, \beta], \mathbb{R}) \cap C([\alpha - r, \beta], \mathbb{R}), (L_g + h)[x] > 0, h(t, x, x_t) \leqslant 0$ for $x \geqslant 0$ and $t \in [\alpha, \beta], g(t, \phi, \varphi), h(t, \phi, \varphi)$ are continuous in t, ϕ and φ where $(t, \phi, \varphi) \in D$. Then the maximum M of x in the interval $[\alpha, \beta]$ cannot be attained anywhere except at the end points α, β.*

Proof. Suppose there exists a $\eta \in (\alpha, \beta)$ such that $x(\eta) = M$. We know that $x'(\eta) = 0$ and $x''(\eta) \leqslant 0$. Then $(L_g)[x]_{t=\eta} \leqslant 0$. In view of the assumption $h(t, x, x_t) \leqslant 0$ for every $x \geqslant 0$, and $t \in [\alpha, \beta]$, we have $(L_g + h)[x]_{t=\eta} \leqslant 0$. This contradicts the assumption of the Lemma. The proof is complete. □

The following theorem is an extension of Theorem 3 (Maximum Principle) in [7, p. 6]

Theorem 2.1. *Suppose $x \in C^2([\alpha, \beta], \mathbb{R}) \cap C([\alpha - r, \beta], \mathbb{R}), (L_g + h)[x] \geqslant 0$ and $h(t, x, x_t) \leqslant 0$ for $x \geqslant 0$ and $t \in [\alpha, \beta], g(\cdot, \phi, \varphi), h(\cdot, \phi, \varphi)$ are C^1 local bounded on $[\alpha, \beta]$. If the maximal value M of x is attained at an interior point of (α, β), then $x(t) \equiv M$ for $t \in (\alpha, \beta)$.*

Proof. For the sake of contradiction, we suppose that there exist $c, d \in (\alpha, \beta)$ such that $x(c) = M$ and $x(d) < M$. Then there are two possible cases (i) $c < d$ and (ii) $c > d$.

(i) $c < d$. From $x(c) = M$ and $x(d) < M$, we know that there exists an interval $(a, b) \subseteq (c, d)$ such that $x(a) = M$ and $x(t) < M$ for all $t \in (a, b)$. Let

$$a_0 \in (\alpha, c), \quad a_\delta = \left(\frac{1}{2}\right)^{\frac{1}{\delta}}(a - a_0) + a_0, \quad b_\delta = 2^{\frac{1}{\delta}}(a - a_0) + a_0, \tag{9}$$

where δ is a positive odd integer and satisfies $\delta \geqslant 3$ and $b_\delta < b$. Then we have

$$\alpha < a_0 < a_\delta < a < b_\delta < b \leqslant d < \beta \tag{10}$$