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Bifurcation analysis of an epidemic model with nonlinear incidence ☆

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ABSTRACT

In this paper, we consider an epidemic model with the nonlinear incidence of a sigmoidal function. By mathematical analysis, it is shown that the model exhibits the bistability and undergoes the Hopf bifurcation and the Bogdanov–Takens bifurcation. By numerical simulations, it is found that the incidence rate can induce multiple limit cycles, and a little change of the parameter could lead to quite different bifurcation structures.

1. Introduction

It has been found that many epidemiological transmissions are affected by the parasite dose to which host is exposed, and relationship between the parasite dose and infection rate is often nonlinear [1,6,11,14]. Regoes et al. [19] introduced a nonlinear incidence into the model which describes the interaction between susceptible hosts S, infected hosts I and free parasites v:

$$\begin{cases} \frac{dS}{dt} = A_0 - dS - g(v)S, \\ \frac{dI}{dt} = g(v)S - (d+\delta)I, \\ \frac{dv}{dt} = \zeta I - uv, \end{cases} \tag{1.1}$$

where A_0 is the birth rate of susceptibles, d is the natural death rate of a host, δ is the per capita infection-related death rate, ζ is the releasing rate of free virus from an infected host, and the rate of infection per host, $g(v)$, is a sigmoidal function:

$$g(v) = \frac{(v/m_0)^k}{1+(v/m_0)^k}, \quad k > 1.$$

Here, m_0 denotes the infectious dose at which 50% of the hosts are infected and k measures the slope of the sigmoidal curve at m_0. It is found in [19] that system (1.1) admits multiple endemic equilibria. Furthermore, by numerical simulations, it is also found that the model has an Allee effect, i.e., the coexistence of a stable uninfected equilibrium and a stable endemic equilibrium in (1.1).

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The purpose of this paper is the following. First, we simplify (1.1) by assuming the dynamics of parasites is substantially faster than that of infected hosts. The advantage of this simplification is that we can use more analytical techniques to reveal the mechanism of the Allee effect and to find more interesting phenomena. Let $u \gg \delta$. If $V = uv$, model (1.1) becomes

$$\begin{cases} \frac{dS}{dt} = A_0 - dS - g(\frac{V}{u})S, \\ \frac{dI}{dt} = g(\frac{V}{u})S - (d+\delta)I, \\ \epsilon\frac{dV}{dt} = \delta(\zeta I - V), \end{cases} \tag{1.2}$$

where $\epsilon = \frac{\delta}{u} \ll 1$. It follows from the quasi-steady-state assumption [19] that system (1.2) can be reduced to:

$$\begin{cases} \frac{dS}{dt} = A_0 - dS - \frac{\beta I^k}{1+\beta I^k}S, \\ \frac{dI}{dt} = \frac{\beta I^k}{1+\beta I^k}S - (d+\delta)I, \end{cases} \tag{1.3}$$

where $\beta = \zeta^k/(um_0)^k$. To be more general, we consider the system:

$$\begin{cases} \frac{dS}{dt} = A_0 - dS - \frac{\beta I^k}{1+\alpha I^k}S, \\ \frac{dI}{dt} = \frac{\beta I^k}{1+\alpha I^k}S - (d+\delta)I, \end{cases} \tag{1.4}$$

where α and β are two independent positive constants and k is a positive parameter. Then the sigmoidal infection rate in (1.4) accommodates the infection force:

$$g(I) = \frac{\beta I}{1+I}, \tag{1.5}$$

which was used by Gumel and Moghadas [9], and the infection force:

$$g(I) = \frac{\beta I^2}{1+\alpha I^2}, \tag{1.6}$$

studied by Ruan and Wang [20]. To our knowledge, studies in literature for epidemic models with the sigmoidal infection force often assume $k = 1$ or $k = 2$. However, parameter k is usually determined empirically as the slope of a Hill plot. From [16], we know that the best fit to data often gives a non-integer value of k. For this reason, in this paper we assume that parameter k is a positive real number and study influences of the general incidence on dynamical behaviors of (1.4) when k varies in $(0,\infty)$. Specifically, we will show that a disease-free equilibrium or an endemic equilibrium is globally stable when $k \leqslant 1$. For $k > 1$, we will derive conditions under which the Allee effect occurs, verify the existence of the Hopf bifurcation and the existence of the Bogdanov–Takens bifurcation, which means that the model exhibits a homoclinic bifurcation. Then we will use parameter α and k as bifurcation parameters, which determine the shape of the sigmoidal incidence, to see how two limit cycles are born and move as the parameters vary. The novelty of our results is that we find that a little change of parameter k could lead to quite different bifurcation structures.

The organization of this paper is as follows. In the next section, we show that the qualitative behaviors of (1.4) is simple for $k \leqslant 1$. Then we analyze conditions for the existence of the Allee effects, the Hopf bifurcation and the Bogdanov–Takens bifurcation in Section 3. Section 4 presents numerical simulations to indicate patterns of dynamical behaviors and bifurcation structures as parameters vary. The paper ends with a brief discussion.

2. Simple dynamics when $k \leqslant 1$

In this section, we show that system (1.4) has the simple dynamics in the sense that an endemic equilibrium is globally stable whenever it exists and a disease-free equilibrium is globally stable if there is no endemic equilibrium. First, we nondimensionalise system (1.4) with the following scaling

$$S = \left(\frac{d+\delta}{\beta}\right)^{\frac{1}{k}}x, \quad I = \left(\frac{d+\delta}{\beta}\right)^{\frac{1}{k}}y, \quad \tau = (d+\delta)t.$$

If t is used to represent τ, we obtain

$$\begin{cases} \frac{dx}{dt} = A - cx - \frac{xy^k}{1+my^k}, \\ \frac{dy}{dt} = \frac{xy^k}{1+my^k} - y, \end{cases} \tag{2.1}$$

where

$$A = \left(\frac{\beta}{d+\delta}\right)^{\frac{1}{k}}\frac{1}{d+\delta}A_0, \quad c = \frac{d}{d+\delta}, \quad m = \frac{\alpha(d+\delta)}{\beta}.$$