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Upwind and midpoint upwind difference methods for time-dependent differential difference equations with layer behavior

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Abstract

In this paper we have considered an important class of time-dependent singularly perturbed convection–diffusion problems with retarded terms which often arise in Computational Neuroscience. We used Taylors series to approximate the retarded terms and the resulting time-dependent singularly perturbed differential equation is approximated using parameters uniform numerical methods based on Euler implicit, upwind and midpoint upwind finite difference schemes. We discretize the continuous problem using implicit Euler scheme in the time direction with a constant step size and the resulting system of equations is approximated using upwind and midpoint upwind difference schemes on a piecewise uniform mesh. We will prove the uniform convergence of these two schemes. Numerical experiments support the convergence results.

Keywords: Singularly perturbed; Differential difference equation; Parabolic PDEs; Convection–diffusion; Neuronal model; Difference scheme; Shishkin mesh; Parameter uniform schemes

1. Introduction

Computational Neuroscience is a highly interdisciplinary subject and various mathematical languages are used to deal with the problems which arise. The possibility of recording single neuron activity induced a great effort to develop adequate mathematical models. In [1–3], one can find various mathematical models for the determination of the behavior of a neuron to random synaptic inputs. The analytical approach for determining the time to first spike from a given initial state using first exit time theory for diffusion process often results in singularly perturbed differential difference equation (DDEs) [4, p. 182],

$$-\frac{\partial f}{\partial t_0}=\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x_0^2}+\left(\mu_{\mathrm{D}}-\frac{x}{\tau}\right)\frac{\partial f}{\partial x_0}+\lambda_{\mathrm{s}} f(x+a_{\mathrm{s}})+\omega_{\mathrm{s}} f(x+i_{\mathrm{s}})-(\lambda_{\mathrm{s}}+\omega_{\mathrm{s}})f, \tag{1.1}$$

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where the first derivative term is because of the exponential decay between two consecutive jumps caused by the input processes. The membrane potential decays exponentially to the resting level with a membrane time constant τ and μ_D and σ are diffusion moments of weiner process characterising the influence of the dendritic synapses on the cell excitability. The reaction terms correspond to the superposition of excitatory and inhibitory inputs, and we can assume that they are poissonian [5]. The excitatory input contributes to the membrane potential by an amplitude a_s with intensity λ_s and similarly the inhibitory input contributes by an amplitude i_s with intensity ω_s. This model makes available time evolution of the trajectories of the membrane potential. Excellent articles for a Neurophysiological and mathematical demonstration of the above model are Stein [6,7], Tuckwell [1,2,8], Tuckwell and Wan [9], Lange and Miura [10].

Moreover, mathematical analysis of singularly perturbed DDEs including its modifications and generalizations has been one of the most common approaches to the theoretical study of neuronal activity [1]. The above application motivates the approximation of singularly perturbed DDEs. In this paper our intention is to elucidate the effect of delays δ and η in a class of singularly perturbed parabolic differential difference equations with layer behavior.

In recent years, many numerical methods have been developed to solve singularly perturbed ordinary differential equations (ODEs) with delay and advance of the type

$$\epsilon^2 u''(x) + b(x)u'(x) + \alpha(x)u(x-\delta) + \beta(x)u(x) + \omega(x)u(x+\eta) = f(x), \quad x \in (0,1). \tag{1.2}$$

Lange and Miura gave a series of papers (to list a few [10–12]) investigating different classes of BVPs of the above type (1.2) by extending the method of matched asymptotic expansions developed for ODEs. Kadalbajoo and Sharma considered different classes of BVPs of type (1.2) giving first-order numerical algorithms based on finite difference schemes and showed the effects of delays on the solution [13]. In [14], Patidar and Sharma constructed a new class of fitted operator finite difference methods for the numerical solution of the above class of BVPs (1.2) and without convection term. Following our pervious works [15–18] we have considered model (2.3) which is a generalization of the above ODE (1.2).

2. Parabolic differential difference equation in x–t plane

In this paper we are concerned with upwind and modified upwind finite difference approximations of a class of second-order singularly perturbed parabolic differential difference equation on the rectangle $D = \Omega \times (0, T]$ in the space–time plane, where $\Omega = (0,1)$ and T is some fixed positive time. Consider the parabolic initial-boundary value problem (I-BVP),

$$\frac{\partial u}{\partial t} - \epsilon^2 \frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x} + \alpha(x)u(x-\delta,t) + \beta(x)u(x,t) + \omega(x)u(x+\eta,t) = f(x,t), \tag{2.3}$$

where $(x,t) \in D$ and is subject to the interval conditions

$$u(x,t) = \phi(x,t) \quad \forall (x,t) \in D_L = \{(x,t) : -\delta \leqslant x \leqslant 0 \text{ and } 0 < t \leqslant T\}, \tag{2.4a}$$

$$u(x,t) = \chi(x,t) \quad \forall (x,t) \in D_R = \{(x,t) : 1 \leqslant x \leqslant 1+\eta \text{ and } 0 < t \leqslant T\} \tag{2.4b}$$

and the initial condition

$$u(x,0) = u_0(x) \quad \forall x \in D_0 = \{(x,0) : x \in \bar{\Omega}\}, \tag{2.5}$$

where $0 < \epsilon \ll 1$ is a constant perturbation parameter and the delay parameters δ and η are assumed to be $O(\epsilon)$. The functions $b(x)$, $\alpha(x)$, $\beta(x)$, $\omega(x)$, $f(x,t)$, $\phi(x,t)$, $\chi(x,t)$ and $u_0(x)$ are smooth, bounded and assumed to be independent of ϵ. It is also assumed that

$$\alpha(x) + \beta(x) + \omega(x) \geqslant d > 0 \quad \forall x \in \bar{\Omega} \tag{2.6}$$

for some constant d [19]. Such problems are addressed as two parameter problem, due to the presence of perturbation (ϵ) and delay (δ, η) parameters. Note that (2.3) contains both delay and advance parameters in the reaction terms and that the special case when $\delta = 0 = \eta$ would reduce the problem to singularly perturbed differential equation which is a one-parameter problem. We are concerned with related class of problems in which delay terms occur in the reaction terms, thereby making it a two parameter problem.