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# Study of boundary value and transmission problems in the Hölder spaces

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ABSTRACT

We present new results on the resolution of singular transmission problems in Hölder spaces completing in this way the work in $L^p$ cases given in [A. Favini, R. Labbas, K. Lemrabet, S. Maingot, Study of the limit of transmission problems in a thin layer by the sum theory of linear operators, Rev. Mater. Comput. 18 (1) (2005) 143–176]. Our approach makes use of the impedance notion operator [M.A. Leontovich, Approximate boundary conditions for the electromagnetic field on the surface of a good conductor. Investigations on radiowave propagation, Moscow Acad. Sci. (Part II) (1948)] which leads to obtain direct and simplified problems. We then use the Dunford calculus and some similar techniques as in [R. Labbas, Problèmes aux limites pour une équation différentielle abstraite de type elliptique, Thèse d'état, Université de Nice, 1987; A. El Haial, R. Labbas, On the ellipticity and solvability of abstract second-order differential equation, Electron. J. Differ. Eq. 57 (2001) 1–18; A. Favini, R. Labbas, S. Maingot, H. Tanabe, A. Yagi, Unified study of elliptic problems in Hölder spaces, C. R. Acad. Sci. Paris Ser. 1341 (2005)] in order to prove existence, uniqueness and maximal regularities results.



## 1. Introduction and hypotheses

Consider the boundary value transmission problem:

$$\begin{cases} (u^\delta)''(x) + Au^\delta(x) = g^\delta(x) \quad \text{in } ]-1,0[\cup]0,\delta[, \\ u^\delta(-1) = f_-, \quad (u^\delta)'(\delta) = f_+^\delta, \\ u^\delta(0^-) = u^\delta(0^+), \quad p_-(u^\delta)'(0^-) = p_+(u^\delta)'(0^+), \end{cases} \tag{1}$$

where $A$ is a closed linear operator of domain $D(A)$, not necessarily dense, embedded in some complex Banach space $E$, $\delta$ is a small positive fixed parameter in $]0,1]$, $g^\delta$ is such that:

$$\begin{cases} g_- = g^\delta|_{[-1,0]} \in C^{2\alpha_0}([-1,0];E), \\ g_+^\delta = g^\delta|_{[0,\delta]} \in C^{2\alpha_0}([0,\delta];E) \end{cases}$$

(with $0 < 2\alpha_0 < 1$). Here $f_-$, $f_+^\delta$ are given in $E$ satisfying some necessary and sufficient conditions which will be specified and $p_-$, $p_+$ are the conductibility positive coefficients of the two bodies $]-1,0[$ and $]0,\delta[$, depending possibly on $\delta$.

Note that the holderianity of $g_-$ and $g_+^\delta$ imply the global holderianity of $g^\delta$ on $[-1,\delta]$ if and only if $g_-(0) = g_+^\delta(0)$.

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Many authors have worked on transmission problems in hilbertian case by using the variational method. Our new approach makes use of the impedance notion characterized by some operator $T_\delta$. This concept will describe precisely the thin layer effect. The specific study of $T_\delta$, in particular its expansion (as $\delta \to 0$), will be important in the forthcoming works since it will allow us to obtain the limit problem.

For the analysis of problem (1), our method consists to solve, at first, the problem:

$$(P_+^\delta) \quad \begin{cases} (u_+^\delta)''(x) + Au_+^\delta(x) = g_+^\delta(x) \text{ in } (0,\delta), \\ u_+^\delta(0) = \psi, \\ (u_+^\delta)'(\delta) = f_+^\delta, \end{cases}$$

($\psi$ is some given element in $E$) in order to define the following impedance operator $T_\delta$:

$$(g_+^\delta, f_+^\delta, \psi) \mapsto T_\delta(g_+^\delta, f_+^\delta, \psi) = (u_+^\delta)'(0),$$

which is intended to express concretely the effect due to the thin layer $(0,\delta)$ and then, we study:

$$(P_-^\delta) \quad \begin{cases} (u_-^\delta)''(x) + Au_-^\delta(x) = g_-(x) \text{ in } (-1,0), \\ u_-^\delta(-1) = f_-, \\ (u_-^\delta)'(0) = \frac{p_+}{p_-} T_\delta(g_+^\delta, f_+^\delta, u_-^\delta(0)) = pT_\delta(g_+^\delta, f_+^\delta, u_+^\delta(0)). \end{cases}$$

We assume in all this work the following ellipticity hypothesis:

$$\varrho(A) \supset [0,+\infty[ \text{ and } \exists C > 0 : \forall \lambda \geqslant 0, \quad \|(A-\lambda I)^{-1}\|_{L(E)} \leqslant \frac{C}{1+|\lambda|}, \tag{2}$$

where $\varrho(A)$ denotes the resolvent set of $A$.

Assumption (2) implies that there exists $\theta_0 \in ]0,\pi/2[$ and $r_0 > 0$ such that:

$$\varrho(A) \supset S_{\theta_0} = \{z \in \mathbb{C}^* : |\arg(z)| \leqslant \theta_0\} \cup \overline{B(0,r_0)} \tag{3}$$

and the estimate in (2) stills true in $S_{\theta_0}$. Let $\gamma$ be the boundary of $S_{\theta_0}$ oriented from $\infty e^{i\theta_0}$ to $\infty e^{-i\theta_0}$.

This paper is organized as follows. In Section 2, we study problem $(P_+^\delta)$, the representation of $u_+^\delta$ is explicitly written by using the functional Dunford calculus. In addition, we give necessary and sufficient compatibility conditions on the data in order to obtain a strict solution $u_+^\delta$ having maximal regularities. Section 3 is devoted to study problem $(P_-^\delta)$.

In Section 4, we illustrate our theory by a model example to which all the results contained in this paper can be applied.

## 2. Study of the problem on $(0,\delta)$

### 2.1. Technical lemmas

Set,

$$\Sigma_{\theta_0} = \{z : |z| \geqslant r_0 \text{ and } \theta_0 \leqslant \arg z \leqslant 2\pi - \theta_0\},$$

and for $z \in \Sigma_{\theta_0}$,

$$\Delta_z(-1,\delta) = \cosh\sqrt{-z}\delta \cosh\sqrt{-z} + p \sinh\sqrt{-z}\delta \sinh\sqrt{-z},$$

$$K_{z,+}^\delta(x,\tau) = \frac{1}{\sqrt{-z}\cosh\sqrt{-z}\delta} \begin{cases} \sinh\sqrt{-z}\tau \cosh\sqrt{-z}(\delta - x) & \text{if } 0 \leqslant \tau \leqslant x, \\ \sinh\sqrt{-z}x \cosh\sqrt{-z}(\delta-\tau) & \text{if } x \leqslant \tau \leqslant \delta. \end{cases}$$

Then, by direct computations, we prove the following lemmas.

**Lemma 1.** *There exists a constant $C_{\theta_0} > 0$ such that for all $z \in \Sigma_{\theta_0}$, we have:*

$$\left|1 + e^{-\sqrt{-z}}\right| \geqslant C_{\theta_0} = 1 - e^{-\frac{\pi}{2\tan\left(\frac{\pi}{2} - \frac{\theta_0}{2}\right)}} > 0.$$

**Remark 1.** In the same way, there exists $C_{\theta_0}$ independent of $\delta \in ]0,1]$ such that:

$$\left|1 + e^{-2\delta\sqrt{-z}}\right| \geqslant C_{\theta_0} > 0 \tag{4}$$

for all $z \in \Sigma_{\theta_0}$.

**Lemma 2.** *For all $\delta \in ]0,1]$ and $z \in \Sigma_{\theta_0}$, we have:*

$$|\Delta_z(-1,\delta)| \geqslant \frac{(C_{\theta_0})^2}{4} \sin\left(\frac{\theta_0}{2}\right) e^{\operatorname{Re}\sqrt{-z}(\delta+1)} > 0.$$