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# Free vibration analysis of circular plates by differential transformation method

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ABSTRACT

This study analyses the free vibrations of circular thin plates for simply supported, clamped and free boundary conditions. The solution method used is differential transform method (DTM), which is a semi-numerical–analytical solution technique that can be applied to various types of differential equations. By using DTM, the governing differential equations are reduced to recurrence relations and its related boundary/regularity conditions are transformed into a set of algebraic equations. The frequency equations are obtained for the possible combinations of the outer edge boundary conditions and the regularity conditions at the center of the circular plate. Numerical results for the dimensionless natural frequencies are presented and then compared to the Bessel function solution and the numerical solutions that appear in literature. It is observed that DTM is a robust and powerful tool for eigenvalue analysis of circular thin plates.



## 1. Introduction

Circular plates are the most critical structural elements in high speed rotating engineering systems such as circular saws, turbine flywheels, rotors, etc. The dynamic characteristics of the plate have a considerable effect on the overall structure performance. When the frequency of the external load matches the natural frequency of the plate, damage or destruction may occur. With this respect, the natural frequencies of the plates have been studied extensively for more than a century. The eigenvalue analysis has been usually carried out by using a variety of weighting function methods [1], including the Galerkin method, the Ritz method and the finite element method. Also, there are voluminous studies, in which the natural frequencies of circular plates are expressed in terms of the Bessel functions [1–3].

The differential transform method (DTM) which is based on the Taylor series expansion was first proposed by Zhou [4] in 1986 for the solution of linear and nonlinear initial value problems that appear in electrical circuits. DTM is a semi-analytical–numerical technique depending on Taylor series and is promising for various types of differential equations. With this technique, it is possible to obtain highly accurate results or exact solutions for differential or integro-differential equations [5]. By using this method, the governing differential equations can be reduced to a recurrence relation and the boundary conditions can be transformed into a set of algebraic equations as in our problem.

In the solution of eigenvalue problems, DTM has been successfully applied in recent studies [6–9]. This study introduces the application of DTM to the solution of eigenvalue problem governing the free vibrations of thin circular plates. There are recent studies on the application of DTM to the dynamical analysis of non-circular plates in literature. Yeh and Jang [10] solved the dynamical problem of the rectangular plate using a hybrid method which combines the finite difference method and the differential transformation method. Malik and Allali [11] derived the characteristic equations of rectangular plates by utilizing

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DTM. Wu et al. [12] investigated the free vibration of circular plate using the generalized differential quadrature rule (GDQR) by applying four boundary/regularity conditions. However, derivation of the frequency equations for circular plates have not been presented yet. This study reveals the frequency equations of free vibrating circular plates. For the first time, an eigenvalue problem with such complex boundary conditions is solved by using DTM. The results of this study are verified by comparing the results of those from the Bessel function and numerical solutions which are available in the studies [1,12]. The results show that DTM is a reliable, fast converging and robust tool for eigenvalue calculation of the circular plate problem.

## 2. Thin plate vibration problem

The governing differential equation of a thin circular plate undergoing free harmonic vibration in a nondimensional form is as follows:

$$\nabla^4 w = \Omega^2 w, \tag{1}$$

where $\nabla^4$ is the biharmonic operator, $w = w(r,\theta)$ is dimensionless deflection, $r$ is dimensionless coordinate along the radial axis of the plate, $\theta$ is dimensionless coordinate along the tangential axis; and $\Omega$ is dimensionless frequency of vibration.

From the classical plate vibration theory, deflection of a circular plate in polar coordinates may be expressed as follows:

$$w = f(r)\cos(m\theta), \tag{2}$$

where $m$ is the integer number of nodal diameters and $f(r)$ is the radial mode function. Substituting Eq. (2) into Eq. (1), the governing differential equation of the circular plate becomes:

$$\frac{d^4f}{dr^4} + \frac{2}{r}\frac{d^3f}{dr^3} - \frac{B}{r^2}\frac{d^2f}{dr^2} + \frac{B}{r^3}\frac{df}{dr} + \frac{A}{r^4}f = \Omega^2 f, \tag{3}$$

where

$$A = m^4 - 4m^2 \quad \text{and} \quad B = 2m^2 + 1. \tag{4}$$

The boundary conditions at the outer edge ($r = 1$) of the circular plate may be one of the following; simply supported, clamped, and free. These conditions may be written in terms of the radial mode function $f(r)$ as follows:

Simply supported:

$$f(r)|_{r=1} = 0, \quad M_r|_{r=1} = -D\left[\frac{d^2f}{dr^2} + \nu\left(\frac{1}{r}\frac{df}{dr} + \frac{m^2}{r^2}f\right)\right] = 0. \tag{5}$$

Clamped:

$$f(r)|_{r=1} = 0, \quad \left.\frac{df}{dr}\right|_{r=1} = 0. \tag{6}$$

Free:

$$M_r|_{r=1} = -D\left[\frac{d^2f}{dr^2} + \nu\left(\frac{1}{r}\frac{df}{dr} + \frac{m^2}{r^2}f\right)\right] = 0, \quad V_r|_{r=1} = \left[\frac{d^3f}{dr^3} + \frac{1}{r}\frac{d^2f}{dr^2} + \left(\frac{m^2\nu - 2m^2 - 1}{r^2}\right)\frac{df}{dr} + \left(\frac{3m^2 - m^2\nu}{r^3}\right)f\right] = 0, \tag{7}$$

where $M_r$ is the radial bending moment, $V_r$ is the effective radial shear force, $D$ is the flexural rigidity and $\nu$ is Poisson's ratio.

It can easily be noticed that, since Eq. (3) is a fourth-order differential equation, four initial conditions are required. One may obtain two of those from the boundary conditions at the outer edge of the circular plate. However, remaining two conditions must be investigated within the regularity conditions at the center of the plate. Wu et al. stated two more boundary conditions at the center of the solid circular plate [12]. If the number of nodal diameter $m$ is assumed, circular plates have symmetric modes for even $m$ and antisymmetric modes for odd $m$ [12]. Therefore, one can obtain regularity conditions at the center of a circular plate ($r = 0$) in terms of the number of nodal diameters as follows:

Antisymmetric case:

$$f(r)|_{r=0} = 0, \quad M_r|_{r=0} = \left.\frac{d^2f}{dr^2}\right|_{r=0} = 0 \quad \text{for } (m = 1, 3, 5, \ldots). \tag{8}$$

Symmetric case:

$$\left.\frac{df}{dr}\right|_{r=0} = 0, \quad V_r|_{r=0} = \left.\frac{d^3f}{dr^3}\right|_{r=0} = 0 \quad \text{for } (m = 0, 2, 4, 6, \ldots). \tag{9}$$

## 3. Differential transform method

Let $f(r)$ be analytic in a domain $R$ and let $r = r_0$ represent any point in $R$. Then, the function $f(r)$ is represented by a power series whose center is located at $r_0$. The differential transform of the $k$th derivative of the function $f(r)$ in one variable is defined as follows: