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Parallel iterative regularization methods for solving systems of ill-posed equations

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ABSTRACT

In this note two parallel iterative regularization methods for finding a minimal-norm solution to a system of ill-posed equations involving the so-called strongly-inverse monotone operators have been investigated. Some applications of the proposed methods are considered and numerical experiments are discussed.

1. Introduction

Various problems of science and engineering can be reduced to finding a solution of a given simultaneous system of operator equations

$$A_i(x) = 0, \quad i = 1, 2, \ldots, N, \tag{1.1}$$

where $A_i : H \to H$ are given possibly nonlinear operators in a real Hilbert space H, and $x \in H$ is an unknown element.

The well-known convex feasibility problem, appearing in many areas, such as optimization theory, image processing, and radiation therapy treatment planning, consists of computing a common solution of equations

$$A_i(x) := x - P_i(x) = 0, \quad i = 1, 2, \ldots, N, \tag{1.2}$$

where P_i are projection operators onto given closed convex sets $C_i \subset H$, $i = 1, 2, \ldots, N$. Problem (1.2) is usually referred to as a common fixed point problem.

A lot of sequential and parallel algorithms for the convex feasibility and common fixed point problems have been proposed. The cyclic projection algorithm, the block-iterative projection algorithm and the explicit iteration algorithm, to name only a few (see, e.g., [4,5,7,8]).

In what follows, we are interested in parallel methods for solving (1.1), where each equation $A_i(x) = 0$ is ill-posed. We refer to [1–3,7,9,11,13] for the general theory of ill-posed problems, and particularly, to [1–3,7] for iterative regularization methods. Unfortunately, these methods, except that one in [7], cannot be used directly for finding a common solution of ill-posed equations. Besides, no numerical examples are available in [7].

The present work is motivated by an interesting idea on regularization for systems of nonlinear equations involving monotone operators [7], and the parallel splitting-up technique for solving nonlinear elliptic problems [10]. We propose a *parallel implicit iterative regularization method* (PIIRM) and a *parallel explicit iterative regularization method* (PEIRM) for solving system (1.1). Moreover, the explicit regularization method suggested in [7] is a particular case of our PEIRM.

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An outline of the remainder of the paper is as follows: In Section 1 we recall some definitions and results that will be used later on in the proof of our main theorems. Section 2 presents our main convergence results, and finally, in Section 3 some applications of the proposed parallel iterative regularization methods are discussed.

Definition 1.1. An operator $A : H \to H$ is said to be demiclosed at u if whenever $u_n \rightharpoonup u$ and $A(u_n) \to v$, then $A(u) = v$.

Hereafter, the symbols $\rightharpoonup$ and $\to$ denote the weak and the convergence in norm, respectively. It is well known (see [6]), that if $T : \Omega \to E$ is a nonexpansive mapping in a uniformly convex Banach space E, and Ω is a nonempty convex subset containing zero of E, then $A := I - T$ is demiclosed at zero.

Definition 1.2. An operator $A : H \to H$ is called c^{-1} – inverse-strongly monotone, if

$$\langle A(x) - A(y), x - y\rangle \geqslant \frac{1}{c}\|A(x) - A(y)\|^2 \quad \forall x, y \in X,$$

where c is some positive constant.

It is proved in [9], that every linear self-adjoint, nonnegative and compact operator in a Hilbert space is inverse-strongly monotone, and the difference between the identity operator and a nonexpansive mapping is an inverse-strongly monotone operator.

Obviously, every inverse-strongly monotone operator is demiclosed at any point. Indeed, let $x_n \rightharpoonup x$ and $A(x_n) \to y$. From the inverse-strong monotonicity of A, it follows

$$c\|A(x_n) - A(x)\|^2 \leqslant \langle A(x_n) - A(x), x_n - x\rangle = \langle A(x_n) - y, x_n - x\rangle - \langle A(x) - y, x_n - x\rangle.$$

The last sum tends to zero because in the first term, $A(x_n) \to y$, while $x_n - x$ is bounded and in the second one, $x_n \rightharpoonup x$. Thus, $A(x) = y$, hence A is demiclosed at x. We also note that every inverse-strongly monotone operator is monotone and not necessarily strongly monotone.

For further consideration, we need the following lemma (see [12]).

Lemma 1.1. *Let $\{a_n\}$ and $\{p_n\}$ be sequences of nonnegative numbers, $\{b_n\}$ be a sequence of positive numbers, satisfying the inequalities*

$$a_{n+1} \leqslant (1 - p_n)a_n + b_n \quad \text{and} \quad p_n < 1 \quad \forall n \geqslant 0,$$

where $\lim_{n\to\infty} \frac{b_n}{p_n} = 0$ and $\sum_{n=1}^{\infty} p_n = +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Theorem 1.2. *Let A_i $(i = 1, 2, \ldots, N)$ be c^{-1}- inverse-strongly monotone operators, defined on the whole space H. Let α_n be a converging to zero sequence of positive numbers. Consider a regularized equation for system (1.1)*

$$\sum_{i=1}^{N} A_i(x) + \alpha_n x = 0. \tag{1.3}$$

Then the following statements hold:

i. *For each $n \in \mathbb{N}$, the regularized equation (1.3) has a unique solution x_n^*.*
ii. *The regularized solution x_n^* converges to the minimal-norm solution x^+ of system (1.1).*
Moreover, there hold estimates:

$$\|x_n^*\| \leqslant \|x^+\|, \tag{1.4}$$

$$\|A_i(x_n^*)\| \leqslant \sqrt{2c\alpha_n}\|x^+\|, \quad i = 1, 2, \ldots, N, \tag{1.5}$$

and

$$\|x_{n+1}^* - x_n^*\| \leqslant \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n}\|x^+\|. \tag{1.6}$$

Proof. See [7, Theorem 2.1]. □

We note that the demiclosedness of operators A_i has been used in the proof of Theorem 1.2.

2. Convergence results

In what follows, we assume that system (1.1) always possesses a solution. Our idea is to solve the regularized equation (1.3) by a modified parallel splitting-up algorithm [10]. The modification is that for each $n \in \mathbb{N}$, only one iteration is performed in solving (1.3).