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Some variants of Hansen–Patrick method with third and fourth order convergence

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ABSTRACT

Based on Hansen–Patrick method [E. Hansen, M. Patrick, A family of root finding methods, Numer. Math. 27 (1977) 257–269], we derive a two-parameter family of methods for solving nonlinear equations. All the methods of the family have third-order convergence, except one which has the fourth-order convergence. In terms of computational cost, all these methods require evaluations of one function, one first derivative and one second derivative per iteration. Numerical examples are given to support that the methods thus obtained are competitive with other robust methods of similar kind. Moreover, it is shown by way of illustration that the present methods, particularly fourth-order method, are very effective in high precision computations.

1. Introduction

One of the most important occurring problems in science and engineering is to find the solutions of the nonlinear equations [1]. The boundary value problems arising in kinetic theory of gases, vibration analysis, design of electric circuits and many applied fields are ultimately reduced to solving such equations. In the present era of advance computers, this problem has gained much importance than ever before.

In this paper, we deal with iterative methods for finding the simple roots of a nonlinear equation $f(x) = 0$, where $f : \mathbb{R} \to \mathbb{R}$, be the continuously differentiable function.

Newton's method is probably the most widely used algorithm for calculating the roots. This method has quadratic convergence and is given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad i = 0, 1, 2, 3, \ldots \tag{1}$$

In order to improve the order of convergence of Newton's method, many third-order methods requiring the evaluation of second derivative have been proposed. For example, Euler's method [2], Halley's method [3], Ostrowski's square root method [4], Super-Halley method [5,6], Hansen–Patrick method [7] are some well-known methods. Recently, some fourth-order methods have also been introduced in [8,9]. These methods are very efficient because of the improvement in order without any addition to the number of evaluations, that is, f, f' and f''.

In this paper, we obtain a two-parameter family of third-order methods based on the Hansen–Patrick family [7]. The new family derives many new and interesting third-order methods in addition to the special cases of the Hansen–Patrick family. Moreover, one particular set of values of the parameters yields a new fourth-order method from our family without adding

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any evaluation to the previously used three evaluations. The efficacy of methods is tested on a number of problems. On comparison with other robust methods, both of our third and fourth order methods behave either similarly or better at least for the examples we consider. Particularly, the fourth-order method is very efficient and even more accurate than the methods of similar kind in the applications by using high precision in computations.

We summarize the contents of this paper. Some basic definitions relevant to the present work are stated in Section 2. In Section 3, we obtain new methods and convergence analysis is carried out to establish third and fourth order convergence of our methods. In Section 4, new methods are tested and compared with other well known methods on a number of difficult problems. Section 5 contains the concluding remarks.

2. Basic definitions

Definition 1. Let $f(x)$ be a real function with a simple root r and let $\{x_i\}_{i\in N}$ be a sequence of real numbers that converges towards r. Then, we say that the order of convergence of the sequence is p, if there exits a $p \in \mathbb{R}^+$ such that

$$\lim_{i\to\infty} \frac{x_{i+1}-r}{(x_i-r)^p} = C, \tag{2}$$

for some $C \neq 0$, C is known as the asymptotic error constant.

If $p = 1, 2$ or 3, the sequence is said to have linear convergence, quadratic convergence or cubic convergence, respectively.

Definition 2. Let $e_i = x_i - r$ is the error in the ith iteration, we call the relation

$$e_{i+1} = Ce_i^p + O\left(e_i^{p+1}\right), \tag{3}$$

as the error equation. If we can obtain the error equation for any iterative method, then the value of p is its order of convergence.

Definition 3. Let θ be the number of new pieces of information required by a method. A 'piece of information' typically is any evaluation of a function or one of its derivatives. The efficiency of the method is measured by the concept of efficiency index [10] and is defined by

$$E = p^{1/\theta}, \tag{4}$$

where p is the order of the method.

3. Development of the methods

In what follows, we construct the new methods from third-order methods involving second derivative. A family of methods of this type called Hansen–Patrick family [7], is expressed as

$$x_{i+1} = x_i - \frac{(\beta+1)}{\beta \pm [1-(\beta+1)L_f(x_i)]^{1/2}}\frac{f(x_i)}{f'(x_i)}, \quad \beta \in \mathbb{R}, \tag{5}$$

where $L_f(x_i) = \frac{f(x_i)f''(x_i)}{f'(x_i)^2}$.

This family includes as particular cases the Ostrowski's square root method, Euler's method, Halley's method, Laguerre's method [4] and, as a limiting case, Newton's method. This family satisfies the following error equation

$$e_{i+1} = \left[\frac{(1-\beta)}{2}A_2^2 - A_3\right]e_i^3 + O(e_i^4), \tag{6}$$

where $A_k = (1/k!)f^{(k)}(r)/f'(r)$, $k = 2, 3, \ldots$

Now considering the Newton-like iterate with parameter α

$$w_i = x_i - \alpha\frac{f(x_i)}{f'(x_i)}, \tag{7}$$

and replacing $f''(x_i)$ by $f''(w_i)$ in (5), our proposed variant of Hansen–Patrick family can be described as

$$x_{i+1} = x_i - \frac{(\beta+1)}{\beta \pm [1-(\beta+1)H_f(x_i)]^{1/2}}\frac{f(x_i)}{f'(x_i)}, \tag{8}$$

with $H_f(x_i) = \frac{f(x_i)f''(w_i)}{f'(x_i)^2}$.