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The global convergence of augmented Lagrangian methods based on NCP function in constrained nonconvex optimization

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ABSTRACT

In this paper, we present the global convergence properties of the primal–dual method using a class of augmented Lagrangian functions based on NCP function for inequality constrained nonconvex optimization problems. We construct four modified augmented Lagrangian methods based on different algorithmic strategies. We show that the convergence to a KKT point or a degenerate point of the original problem can be ensured without requiring the boundedness condition of the multiplier sequence.

1. Introduction

Consider the following inequality constrained nonlinear optimization problem:

$$(P)\quad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \geqslant 0, \quad i = 1, \ldots, m, \\ & x \in X, \end{array}$$

where f and each $g_i : \mathbb{R}^n \to \mathbb{R}$ are all continuously differentiable functions, and X is a nonempty closed convex set in $\mathbb{R}^n$. Notice that f and each $-g_i$ are not necessarily convex.

The first augmented Lagrangian method was proposed by Hestenes [14] and Powell [32] in order to eliminate the duality gap between an equality constrained problem and its Lagrangian dual problem. Later, Rockafellar [34,35] extended this method to deal with inequality constraints. Since then, various modified augmented Lagrangian methods have been proposed. The strong duality properties and exact penalization of different types of augmented Lagrangians or nonlinear Lagrangians have been studied by many researchers (see e.g., [15–17,20,28,30,37–39]). The existence of a saddle point of certain augmented or nonlinear Lagrangian functions has been investigated in [20–22,36,39,41]. Local convergence results of augmented Lagrangian methods were studied in [10,12,26,27,30]. The global convergence of Rockafellar's augmented Lagrangian methods for nonconvex constrained problems were established in [3,13,29,42] under a restrictive assumption that the sequence of multipliers generated by the algorithms is bounded. Modified augmented Lagrangian methods for nonconvex constrained problems have been proposed to ensure the global convergence without appealing to the boundedness assumption of the multiplier sequence (see [1,2,5,8,9,19]). Global convergence of augmented Lagrangian methods for convex programming has been also studied in [4,18,31,34,40]. Convergence to a global optimal solution of modified augmented Lagrangian methods was established in [23–25] for nonconvex constrained global optimization problems without requiring the boundedness of the multiplier sequence.

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Recently, a augmented Lagrangian function for (P) was proposed in [33] based on Fishcher–Burmeister NCP function. Good numerical behavior of this augmented Lagrangian function was shown in [33] by comparing with the other known ones. We now consider a general class of augmented Lagrangian functions based on NCP Function. Let,

$$L(x,\lambda,c) = f(x) - \frac{1}{c}\sum_{i=1}^{m} W(cg_i(x),\lambda_i), \tag{1}$$

where $c > 0$, $x \in X$, $\lambda = (\lambda_1,\ldots,\lambda_m)^T \geqslant 0$, and the function $W : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$W(s,t) = st + \int_0^s \phi(u,t)du. \tag{2}$$

Here, the function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is assumed to satisfy the following conditions:

(A1) $\phi(s,t) = 0 \iff s \geqslant 0,\ t \geqslant 0,\ st = 0$;
(A2) $\phi(s,t) + t \geqslant 0,\ \forall s \in \mathbb{R},\ t \in \mathbb{R}_+$, where $\mathbb{R}_+ = [0,\infty)$;
(A3) $\lim_{s\to+\infty}\phi(s,t) = -t$, $\lim_{s\to-\infty}\phi(s,t) = +\infty$, $\forall t \in \mathbb{R}_+$.

The function ϕ satisfying condition (A1) is called an NCP function. Examples of ϕ that satisfy conditions (A1)–(A3) include the min-function $\phi^{\min}(s,t) = \frac{1}{2}\left(\sqrt{(s-t)^2} - s - t\right) = -\min\{s,t\}$ and the Fischer–Burmeister function $\phi^{\mathrm{FB}}(s,t) = \sqrt{s^2+t^2} - s - t$.

We note that

$$st + \int_0^s \phi^{\min}(u,t)du = -\frac{1}{2}[\min\{0,s-t\}]^2 + \frac{1}{2}t^2,$$

the augmented Lagrangian of Rockafellar [34,35] is a special case of $L(x,\lambda,c)$ when setting $\phi(s,t) = \phi^{\min}(s,t)$ in (2). When taking $\phi(s,t) = \phi^{\mathrm{FB}}(s,t)$ in (2), $L(x,\lambda,c)$ reduces to the augmented Lagrangian function given in [33]. Local convergence results of augmented Lagrangian methods using $L(x,\lambda,c)$ associated with $\phi(s,t) = \phi^{\mathrm{FB}}(s,t)$ were studied in [33].

One purpose of this paper is to study global convergence properties of augmented Lagrangian methods based on $L(x,\lambda,c)$. Four different algorithmic strategies are considered to circumvent the boundedness condition of the multipliers in the convergence analysis for basic primal–dual method. We first show that under weaker conditions, the augmented Lagrangian method using safeguarding strategy converges to a KKT point or a degenerate point of the original problem. The convergence properties of the augmented Lagrangian method using conditional multiplier updating rule is then presented. Finally, we investigate the use of penalty parameter updating criteria and normalization of the multipliers in augmented Lagrangian methods.

The paper is organized as follows. In Section 2, the convergence properties of the basic primal–dual method under the standard assumptions. The modified augmented Lagrangian method with safeguarding is investigated in Section 3. In Section 4, we establish the convergence results of the augmented Lagrangian method with the conditional multiplier updating. The use of penalty parameter updating criteria and normalization of multipliers are discussed in Section 5. Some preliminary numerical results are reported in Section 6. Finally, some concluding remarks are given in Section 7.

2. Basic primal–dual scheme

In this section, we present the basic primal–dual scheme based on the above class of augmented Lagrangians $L(x,\lambda,c)$ and discuss its convergence to a KKT point or a degenerate point under standard conditions.

Define $h(x,\lambda,c) = (h_1(x,\lambda,c),\ldots,h_m(x,\lambda,c))^T$ with

$$h_i(x,\lambda,c) = (1/c)\phi(cg_i(x),\lambda_i), \quad i = 1,\ldots,m. \tag{3}$$

Algorithm 1. Basic primal–dual method

Step 0. (Initialization) Select two positive sequences $\{c_k\}_{k=0}^{\infty}$ and $\{\epsilon_k\}_{k=0}^{\infty}$ with $\epsilon_k \to 0$ as $k \to \infty$. Choose $\lambda^0 \geqslant 0$. Set $k = 0$.
Step 1. (Relaxation problem) Compute an $x^k \in X$ such that:

$$\|\mathscr{P}[x^k - \nabla_x L(x^k,\lambda^k,c_k)] - x^k\| \leqslant \epsilon_k, \tag{4}$$

where $\mathscr{P}$ is the Euclidean projection operator onto X.
Step 2. (Multiplier updating) Compute

$$\lambda^{k+1} = \lambda^k + c_k h(x^k,\lambda^k,c_k). \tag{5}$$

Step 3. Set $k = k+1$, go to Step 1.