Applied Mathematics and Computation 212 (2009) 135–144

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A new system of variational inclusions involving $H(\cdot,\cdot)$-accretive operator in Banach spaces

Yun-Zhi Zou, Nan-Jing Huang *

Department of Mathematics, Sichuan University, Wangjiang Road 27, Chengdu, Sichuan 610064, PR China

ARTICLE INFO

Keywords:
$H(\cdot,\cdot)$-accretive operator
Resolvent operator
A system of variational inclusions
Iterative algorithm
Convergence

ABSTRACT

In this paper, we introduced and study a new class of system of variational inclusions involving $H(\cdot,\cdot)$-accretive operator in Banach spaces. By using the resolvent operator technique associated with $H(\cdot,\cdot)$-accretive operator, we prove the existence of the solution for the system of inclusions and develop a step-controlled iterative algorithm to approach the unique solution.

1. Introduction

Variational inclusions, as the generalization of variational inequalities, have been widely studied in recent years. Many efficient ways have been studied to find solutions for variational inclusions. Those methods include projection method and its various forms, linear approximation, descent and Newton's method, and the method based on auxiliary principle technique etcetera. The method based on the resolvent operator technique is a generalization of projection method and has been widely used to solve variational inclusions. For details, we refer to [1–7,13–19,21–23,25–27,29,30,32,34–36] and the references therein.

It is known that accretivity of the underlying operator plays indispensable roles in the theory of variational inequality and its generalizations. In 2001, Huang and Fang [20] were the first to introduce generalized m-accretive mapping and give the definition of the resolvent operator for generalized m-accretive mappings in Banach spaces. They also showed some properties of the resolvent operator for generalized m-accretive mappings. Recently, Fang and Huang, Lan, Cho and Verma investigated many generalized operators such as H-monotone [8], H-accretive [10], (H,η)-accretive [11], (H,η)-monotone [9,12], (A,η)-accretive mappings [28]. Very recently, Zou and Huang [37] introduced and studied a new class of $H(\cdot,\cdot)$-accretive operators in Banach spaces which provided unifying work for the H-accretive, (H,η)-accretive, (A,η)-accretive mappings in Banach spaces. They showed some properties of the resolvent operator for the $H(\cdot,\cdot)$-accretive operator and obtained an application for solving variational inclusions in Banach spaces. They also gave some examples to illustrate their results.

Inspired and motivated by their excellent work, in this paper, we introduce and discuss a new class of system of variational inclusions involving the $H(\cdot,\cdot)$-accretive operator in Banach spaces. By using the resolvent operator technique associated with $H(\cdot,\cdot)$-accretive operator due to Zou and Huang [37], we prove the existence of the solution for the system of inclusions and develop a step-controlled iterative algorithm to approach the unique solution. The results presented in this paper improve and generalize many known results in the literature.

2. Preliminaries

Let X be a real Banach space with dual space X^*, $\langle\cdot,\cdot\rangle$ be the dual pair between X and X^*, and 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

* Corresponding author.
E-mail address: nanjinghuang@hotmail.com (N.-J. Huang).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2009.02.007

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-1} J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex. In the sequel, we always assume that X is a real Banach space such that J_q is single-valued. If X is a Hilbert space, then J_2 becomes the identity mapping on X.

The modulus of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup \left\{ \frac{1}{2} \|x + y\| + \|x - y\| - 1 : \|x\| \leqslant 1, \|y\| \leqslant t \right\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0.$$

X is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_X(t) \leqslant ct^q, \quad q > 1.$$

Note that J_q is single-valued if X is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [33] proved the following result.

Lemma 2.1. *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$,*

$$\|x + y\|^q \leqslant \|x\|^q + q\langle y, J_q(x) \rangle + c_q \|y\|^q.$$

Definition 2.1. Let $A, B : X \to X$ be two single-valued mappings and $H, \eta : X \times X \to X$ be two single-valued mappings.

(i) A is said to be accretive if

$$\langle Ax - Ay, J_q(x - y) \rangle \geqslant 0, \quad \forall x, y \in X;$$

(ii) A is strictly accretive if A is accretive and

$$\langle Ax - Ay, J_q(x - y) \rangle = 0,$$

if and only if $x = y$;

(iii) $H(A, \cdot)$ is said to be α-strongly accretive with respect to A if there exists a constant $\alpha > 0$ satisfying

$$\langle H(Ax, u) - H(Ay, u), J_q(x - y) \rangle \geqslant \alpha \|x - y\|^q, \quad \forall x, y, u \in X;$$

(iv) $H(\cdot, B)$ is said to be β-relaxed accretive with respect to B if there exists a constant $\beta > 0$ such that

$$\langle H(u, Bx) - H(u, By), J_q(x - y) \rangle \geqslant -\beta \|x - y\|^q, \quad \forall x, y, u \in X;$$

(v) $H(\cdot, \cdot)$ is said to be ρ-Lipschitz continuous with respect to A if there exists a constant $\rho > 0$ such that

$$\|H(Ax, u) - H(Ay, u)\| \leqslant \rho \|x - y\|, \quad \forall x, y, u \in X;$$

(vi) A is said to be γ-Lipschitz continuous if there exists a constant $\gamma > 0$ such that

$$\|Ax - Ay\| \leqslant \gamma \|x - y\|, \quad \forall x, y \in X;$$

(vii) $\eta(x, \cdot)$ is said to be strongly accretive with respect to $H(A, B)$, if there exists a constant ξ such that

$$\langle \eta(x, u) - \eta(y, u), J_q(H(Ax, Bx) - H(Ay, By)) \rangle \geqslant \xi \|x - y\|^q, \quad \forall x, u, y \in X.$$

In a similar way to (v) and (vii), we can define the Lipschitz continuity of the mapping H with respect to B and the strong accretivity of $\eta(\cdot, y)$ with respect to $H(A, B)$.

Definition 2.2. Let $\eta : X \times X \to X$ and $H, A : X \to X$ be three single-valued mappings. Let $M : X \to 2^X$ be a set-valued mapping.

(i) η is said to be τ-Lipschitz continuous if there exists a constant τ such that

$$\|\eta(x, y)\| \leqslant \tau \|x - y\|, \quad \forall x, y \in X;$$

(ii) M is said to be accretive if

$$\langle u - v, J_q(x - y) \rangle \geqslant 0, \quad \forall x, y \in X, u \in Mx, v \in My;$$

(iii) M is said to be η-accretive if

$$\langle u - v, J_q(\eta(x, y)) \rangle \geqslant 0, \quad \forall x, y \in X, u \in Mx, v \in My;$$