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Path embedding in star graphs

Ming-Chien Yang

Department of Knowledge Management, Aletheia University, Tainan County 721, Taiwan, ROC

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ABSTRACT

The star graph interconnection network has been introduced as an attractive alternative to the hypercube network. In this paper, we consider the path embedding problem in star graphs. Assume that $n \geqslant 4$. We prove that paths of all even lengths from $d(x,y)$ to $n!-2$ can be embedded between two arbitrary vertices x and y from the same partite set in the n-dimensional star graph. In addition, paths of all odd lengths from $d(x,y)$ to $n!-1$ can be embedded between two arbitrary vertices x and y from different partite sets in the n-dimensional star graph except that if x and y are adjacent, there is no path of length 3 between them. The result is optimal in the sense that paths of all possible lengths are found in star graphs.

1. Introduction

In parallel processing systems, processors are connected based on an interconnection network. It can be represented by a graph with vertices corresponding to processors and edges corresponding to links. We use graphs and interconnection networks interchangeably in this paper. *Graph embedding* is an important factor to evaluate an interconnection network. Embedding a guest graph G into a host graph H is a mapping from each vertex of G to a vertex of H and a mapping from each edge of G to a path of H. Graph embedding can be used to model the simulation of one guest graph by one host graph. The ring [3,4,14,15,25,26,29,31], path [8,9,28,30], mesh [32], and tree [6,17,18] embeddings into various topologies have earned a lot of interests. In particular, Fan et al. studied the path embedding problem in crossed cubes [8]. They found that paths of all lengths from $\lceil \frac{n+1}{2} \rceil$ to $2^n - 1$ can be embedded between two arbitrary vertices in the n-dimensional crossed cube.

Among the various interconnection networks, the star graph has attracted much attention. The desirable properties of the star graph include recursive structure, vertex transitive, edge transitive, sublogarithmic degree and diameter, and maximal fault-tolerance [1,2]. Efficient algorithms were proposed, including parallel routing [7], multicasting [10], and broadcasting [22,27] algorithms. The diameter and fault diameter of star graphs were studied in [2,19,23,24]. Works on the path and cycle embedding problems in star graphs can also be found in the literature. Hsieh et al. studied the problem of hamiltonian-laceability [11], and gave fault-tolerant longest path embedding in star graphs [12,13]. Lin et al. found mutually independent hamiltonian paths in star graphs [21]. In [20], Li provided fault-free cycle embedding with edge failures in star graphs. Xu et al. further studied the fault-tolerant edge-bipancyclic property of star graphs in [29].

In this paper, we study the path embedding problem in star graphs. A graph G is *bipartite* if the vertex set of G is the union of two disjoint sets such that every edge contains one vertex from each set. The star graphs have been shown to be bipartite [16]. Therefore, in a star graph, there are no path of any odd length between two vertices from the same partite set and no path of any even length between two vertices from different partite sets. The contributions of this paper are as follows. Assume that $n \geqslant 4$. Let x and y be two arbitrary distinct vertices of the n-dimensional star graph. If x and y are adjacent, for an arbitrary odd integer l with $5 \leqslant l \leqslant n!-1$, there exists a path of length l between x and y. If x and y belong to different partite sets and $d(x,y) \geqslant 3$, there exists a path of length l between x and y for an arbitrary odd integer l with $d(x,y) \leqslant l \leqslant n!-1$. If x

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E-mail address: mcyang@mail.au.edu.tw

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and y belong to the same partite set, there exists a path of length l between x and y for an arbitrary even integer l with $d(x,y) \leqslant l \leqslant n! - 2$. Since the n-dimensional star graph has $n!$ vertices and no cycle of length 4, we have found paths of all possible lengths between two arbitrary distinct vertices of it.

2. Star graphs and basic properties

Letting G be a simple undirected graph, we use $V(G)$ and $E(G)$ to denote the sets of vertices and edges of G, respectively. Also, we use $|V(G)|$ and $|E(G)|$ to denote the numbers of vertices and edges of G, respectively. A *path*, denoted by $\langle v_1, v_2, \ldots, v_k \rangle$, is defined as a sequence of vertices where two successive vertices are adjacent in G. A *cycle* is a path that begins and ends with the same vertex.

We denote the set $\{1, 2, \ldots, n\}$ by $\langle n \rangle$, where n is a positive integer. A *permutation* on $\langle n \rangle$ is a sequence of n distinct elements of $\langle n \rangle$. A number u_i in a permutation $u_1u_2 \cdots u_n$ on $\langle n \rangle$ is in its "correct" position if $u_i = i$ for some $1 \leqslant i \leqslant n$. The identity permutation $12 \cdots n$ is denoted by ε, in which each number is in its correct position. A $\pi[1,i]$ transposition on a permutation $u = u_1u_2 \cdots u_n$ is to swap the leftmost number with the ith number in u, i.e., $\pi[1,i](u) = u_iu_2 \cdots u_{i-1}u_1u_{i+1} \cdots u_n$ for some $2 \leqslant i \leqslant n$.

An n-dimensional star graph, denoted by S_n, is defined as follows. The vertex set of S_n is $\{u \mid u \text{ is a permutation on } \langle n \rangle\}$. The edge set of S_n is $\{(u, \pi[1,i](u)) |$ for every $2 \leqslant i \leqslant n\}$. That is, two vertices of S_n are adjacent if they can be obtained from each other by swapping the leftmost number with one of the other $n-1$ numbers. Therefore, S_n has $n!$ vertices, and is an $(n-1)$-regular graph. Furthermore, it has been shown that S_n is vertex transitive and edge transitive [2]. S_1, S_2, and S_3 are a vertex, an edge, and a cycle of length 6, respectively. S_4 is shown in Fig. 1. We observe that S_4 has no cycles of length 4. In fact, S_n has no cycles of length 4 for any $n \geqslant 1$. A cycle of length l is called an l-cycle for any $l \geqslant 3$.

Theorem 1. *The star graph has no 4-cycle.*

Proof. Trivially, S_1, S_2, and S_3 have no 4-cycle. Suppose that $n \geqslant 4$ and $\langle x, y, u, v \rangle$ is a 4-cycle of S_n. Let $x = x_1x_2 \cdots x_n$. Without loss of generality, we may assume that $y = \pi[1,2](x) = x_2x_1x_3 \cdots x_n$ and $u = \pi[1,3](y) = x_3x_1x_2 \cdots x_n$. Then, either $v = \pi[1,2](u) = x_1x_3x_2 \cdots x_n$ or $v = \pi[1,i](u) = x_ix_1x_2 \cdots x_{i-1}x_3x_{i+1} \cdots x_n$ for some $4 \leqslant i \leqslant n$. In either case, v is not adjacent to x, which is a contradiction. □

S_n contains n disjoint copies of S_{n-1}. More precisely, for some $2 \leqslant i \leqslant n$, an *i-partition* on S_n partitions S_n into n copies of $(n-1)$-*dimensional substars* (or $(n-1)$-*substars*), denoted by S_n^j, where $V(S_n^j) = \{u_1u_2 \cdots u_n | u_1u_2 \cdots u_n$ is a vertex of S_n with $u_i = j\}$ for each $1 \leqslant j \leqslant n$. For instance, we apply the 2-partition on S_4. Then, the subgraph induced by {1423, 1432, 3412, 3421, 2413, 2431} is a 3-substar, S_4^4. Given an arbitrary vertex x of an $(n-1)$-substar, a neighbor of x is called an *in-neighbor* of x if they are in the same $(n-1)$-substar. On the other hand, a neighbor of x is called the *out-neighbor* of x if they are in different $(n-1)$-substars. For example, applying the 3-partition on S_5, 32451 is a vertex of S_5^4, and it has in-neighbors 23451, 52431, and 12453. On the other hand, 42351 is the out-neighbor of 32451.

Lemma 1. *Suppose that x is a vertex of S_n^r having the out-neighbor $u \in V(S_n^s)$ after applying an i-partition for some $1 \leqslant r, s \leqslant n$ and $2 \leqslant i \leqslant n$. For each $k \in \langle n \rangle - \{r, s\}$, there exists an in-neighbor of x such that its out-neighbor is a vertex of S_n^k.*

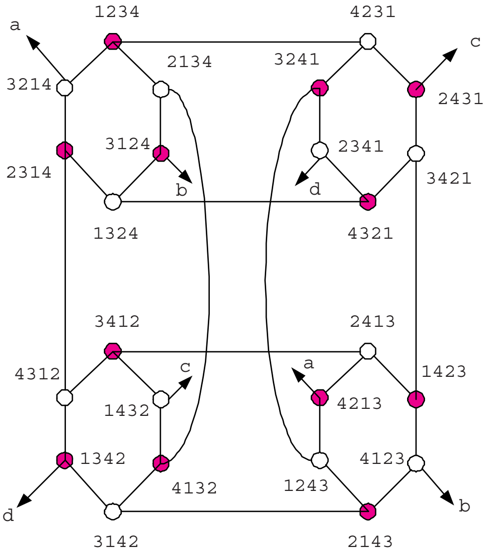

Fig. 1. Illustration of S_4.