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Integral forms of sums associated with harmonic numbers

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ABSTRACT

We use integral identities to establish a relationship with sums that include harmonic numbers, more over we obtain some closed forms of finite binomial sums. In particular cases, we establish some identities for harmonic numbers.

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1. Introduction

The generalised Harmonic number in power α is defined as

$$H_n^{(\alpha)} = \sum_{r=1}^{n} \frac{1}{r^\alpha}.$$

The nth Harmonic number

$$H_n^{(1)} = H_n = \int_{t=0}^{1} \frac{1-t^n}{1-t}dt = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi(n+1),$$

where γ denotes the Euler–Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{r=1}^{n} \frac{1}{r} - \log(n)\right) = -\psi(1) \approx 0.5772156649015328606065 12\ldots$$

Harmonic numbers are closely related the Riemann ζ-function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} (1-p^{-s})^{-1},$$

where the product is over all primes p. It is well known that $H_n^{(1)} = \frac{|s(n+1,2)|}{n!}$ where $s(n,k)$ are Stirling numbers of the first kind, defined by

$$(w)_n = \prod_{r=1}^{n} (w+1-r) = \sum_{k=0}^{n} s(n,k)w^k.$$

If we let $\Omega(n,k) = |s(n,k)|$, be the unsigned Stirling numbers of the first kind, which counts the permutations of n elements that are the product of k disjoint cycles, then it is known that

$$\zeta(k+1) = \sum_{n=k}^{\infty} \frac{\Omega(n,k)}{n\,n!}.$$

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The evaluation of series with Harmonic numbers, or series which sum to Harmonic numbers dates back to the time of Euler. By a process of exprapolation Euler discovered that $\sum_{r=1}^{\infty}\frac{H_{r-1}^{(p)}}{r^m}$ can be evaluated in terms of Riemann ζ-functions when $p + m = 0$. Euler gave, amongst many other identities, the beautiful result

$$\sum_{r=1}^{\infty}\frac{H_r^{(1)}}{(r+1)^2}=\zeta(3)$$

and its generalisation

$$\sum_{r=1}^{\infty}\frac{H_r^{(1)}}{r^m}=\frac{1}{2}(m+2)\zeta(m+1)-\sum_{r=1}^{m-2}\zeta(m-r)\zeta(r+1)\quad \text{for } m\in\mathbb{N}\setminus\{1\}.$$

Since then many other results have been obtained. The sum

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}\binom{n}{k}}{k}=H_n^{(1)}$$

is well known and appears in a number of problems related to random allocations and theory of records. Adamchik [1] for example obtained

$$\sum_{k=1}^{n}\frac{H_k^{(1)}}{k}=(H_n^{(1)})^2+H_n^{(1)}$$

from which it is possible to extract the result

$$\sum_{k=0}^{n}\frac{(-1)^{k}\binom{n}{k}}{2(k+1)}\left[\left(H_{k+1}^{(1)}\right)^2+H_{k+1}^{(1)}\right]=\frac{1}{(n+1)^3}.$$

Flajolet and Sedgewick [5] also got results of the type

$$6\sum_{k=1}^{n}\frac{(-1)^{k+1}\binom{n}{k}}{k^3}=\left(H_n^{(1)}\right)^3+3H_n^{(1)}H_n^{(2)}+2H_n^{(3)}. \tag{1.1}$$

There are many papers dealing with identities of harmonic numbers, see for example, [15,3].

The idea of derivative operators of binomial coefficients allows one to handle harmonic number identities by reducing them to a hypergeometric problem. This technique was often used by Andrews, in particular in Andrews and Uchimura [2] there is a statement by Askey, pointing out that it was Newton [9], the first person, to see that the partial sums of harmonic series arise from differentiation of a product.

In this paper we will use the idea of the consecutive derivative operator of binomial coefficients to give integral representation for series of the form

$$\sum_{n=0}^{p}t^n n^m\binom{p}{n}Q^{(q)}(a,j)=\int f(\cdot;x)dx,$$

where $Q(a,j)=\binom{an+j}{j}^{-1}$ is the binomial coefficient and $Q^{(q)}(a,j)=\frac{d^q}{dj^q}[Q(a,j)]$ is the qth derivative of the binomial coefficient. In particular cases of the parameter values (a,j,p,q,t) we will derive some identities for sums of harmonic numbers. In other cases where the sums cannot be expressed in closed form the integral identity may be used to obtain bounds on the sum, this will not be explored here.

Binomial coefficients play an important role in many areas of mathematics, including number theory, statistics and probability. The binomial coefficient is defined as

$$\binom{z}{w}=\frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$$

for z and w non-negative integers, where $\Gamma(x)$ is the Gamma function.

The representation of sums in closed form can in some cases be achieved through a variety of different methods, including, integral representations, transform techniques, Riordan arrays and the W–Z method. The interested reader is referred to the works of Petkovsek et al. [10], Sofo [11–13], Egorychev [4] and Merlini et al. [8].

The following lemmas will be useful in the main results of this paper.

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