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# Two systems of two-component integrable equations: Multiple soliton solutions and multiple singular soliton solutions

Abdul-Majid Wazwaz

*Department of Mathematics, Saint Xavier University, Chicago, IL 60655, United States*

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ABSTRACT

In this work, we study two systems of two-component integrable equations. The Cole–Hopf transformation and Hirota's bilinear method are applied to emphasize the integrability of each system. Multiple soliton solutions and multiple singular soliton solutions are formally derived.



## 1. Introduction

For completely integrable evolution equations, many distinct methods namely, the generalized symmetry method [1–6], the inverse scattering method, the Bäcklund transformation method, and the Hirota bilinear method [7–17] were thoroughly used to derive the multiple soliton solutions of these equations. Hirota's bilinear method is rather heuristic and possesses significant features that make it practical for the determination of multiple soliton solutions [18–30] for a wide class of nonlinear evolution equations in a direct method. Moreover, the tanh method [31–34] was used to determine single soliton solution. The computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations.

As stated before, the classification of integrable equations is usually examined by many distinct approaches. In [1–4], the complete classification of coupled symmetric KdV-type equations and Burgers-type equations, that possess higher generalized symmetries, was thoroughly addressed. Foursov [1] applied the generalized symmetry method to derive several systems, where 11 of these systems were previously unknown classes of integrable equations of the coupled potential KdV and modified KdV-type equations.

Two integrable coupled potential KdV equations and modified KdV-type equations [1], termed by D. System (11) and I. System (6), given respectively by

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + 2u_x^2 + v_x^2, \\ v_t &= \frac{1}{2}u_{xxx} + \frac{1}{2}v_{xxx} + 2v_x^2 + u_x^2 \end{aligned} \tag{1}$$

and

$$\begin{aligned} u_t &= u_{xxx} + 3uvu_x + u^2v_x, \\ v_t &= v_{xxx} + 3uvv_x + v^2u_x \end{aligned} \tag{2}$$

will be selected for further study in this work. The tanh method [31–34] will be first used to determine the relation between the solutions $u(x,t)$ and $v(x,t)$. The Cole–Hopf transformation method combined with Hirota's bilinear method [7–17,23–30]

*E-mail address:* wazwaz@sxu.edu

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will be used next. Our goal from applying these methods is to construct multiple regular soliton solutions and multiple singular soliton solutions. Hirota and Ito in [7] examined the phenomena of two solitons near resonant state, two solitons at the resonant state, and two solitons after colliding with each other. It was proved in [7] that two solitons become singular after colliding with each other, in the sense that regular solitons with $\text{sech}^2$ profiles are transmitted into singular solitons with $\text{cosech}^2$ profiles through the interaction.

In what follows we highlight briefly the main steps of Hirota's bilinear method [7–17] that will be used in this work. The other two methods will be presented in the forthcoming sections.

## 2. Hirota's bilinear method

We first substitute

$$u(x,t) = e^{kx-ct} \tag{3}$$

into the linear terms of the equation under discussion to determine the dispersion relation between $k$ and $c$. We then substitute the single soliton solution

$$u(x,t) = R\left(\arctan\left(\frac{f(x,y,t)}{g(x,y,t)}\right)\right)_x, \tag{4}$$

$$u(x,t) = R(\ln f(x,t))_x = R\frac{f_x}{f}, \tag{5}$$

or

$$u(x,t) = R(\ln f(x,t))_{xx} = R\frac{ff_{xx} - f_x^2}{f^2} \tag{6}$$

into the equation under discussion, where the auxiliary function $f(x,t)$ is given by

$$f(x,t) = 1 + f_1(x,t) = 1 + e^{\theta_1}, \tag{7}$$

where

$$\theta_i = k_i x - c_i t, \quad i = 1, 2, \ldots, N \tag{8}$$

and solving the resulting equation to determine the numerical value for $R$. The $N$-soliton solutions can be obtained following the main steps:

(i) For dispersion relation, we use

$$u(x,t) = e^{\theta_i}, \quad \theta_i = k_i x - c_i t. \tag{9}$$

(ii) For single soliton, we use

$$f(x,t) = 1 + e^{\theta_1}. \tag{10}$$

(iii) For two-soliton solutions, we use

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}. \tag{11}$$

(iv) For three-soliton solutions, we use

$$f(x,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} + a_{13}e^{\theta_1+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3}. \tag{12}$$

However, for the multiple singular soliton solutions, we follow the following steps:

(i) For dispersion relation, we use

$$u(x,t) = e^{\theta_i}, \quad \theta_i = k_i x - \omega_i t. \tag{13}$$

(ii) For single soliton, we use

$$f(x,t) = 1 - e^{\theta_1}. \tag{14}$$

(iii) For two-soliton solutions, we use

$$f(x,t) = 1 - e^{\theta_1} - e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}. \tag{15}$$

(iv) For three-soliton solutions, we use

$$f(x,t) = 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} + a_{13}e^{\theta_1+\theta_3} - b_{123}e^{\theta_1+\theta_2+\theta_3}. \tag{16}$$