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Some presentations, connections and series expansions for the generalized Voigt functions

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ABSTRACT

The principal object of this paper is to present a natural further step toward the mathematical properties and presentations concerning the generalized Voigt functions. Recurrence relations, connections, series expansions and integral representations involving classical functions of mathematical physics and hypergeometric series for these functions are established. Some particular cases and consequences of our main results are also considered.

1. Introduction

The familiar Voigt functions [2]

$$K(x,y) = \pi^{-1/2}\int_0^\infty \exp\left[-yt - \frac{1}{4}t^2\right]\cos(xt)dt, \tag{1.1}$$

$$L(x,y) = \pi^{-1/2}\int_0^\infty \exp\left[-yt - \frac{1}{4}t^2\right]\sin(xt)dt \quad (-\infty < x < \infty, y > 0) \tag{1.2}$$

occur frequently in a wide variety of physical problems such as astrophysical spectroscopy, neutron physics, plasma physics and statistical communication theory, as well as some areas in mathematical physics and engineering associated with multidimensional analysis of spectral harmonic.

For the Bessel function $J_v(z)$ of order v defined by

$$J_v(z) = \sum_{m=0}^{\infty}\frac{(-1)^m(z/2)^{v+2m}}{m!\Gamma(v+m+1)}, \quad |z| < \infty, \tag{1.3}$$

it is will known that

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\cos(z) \quad \text{and} \quad J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin(z). \tag{1.4}$$

Motivated by the relationships (1.4) Srivastava and Miller [11] introduced and studied rather systematically a unification (and generalization) of the Voigt functions $K(x,y)$ and $L(x,y)$ in the form

$$V_{\mu,v}(x,y) = \left(\frac{x}{2}\right)^{1/2}\int_0^\infty t^\mu \exp\left[-yt - \frac{1}{4}t^2\right]J_v(xt)dt \quad (x,y \in R^+, Re(\mu+v) > -1), \tag{1.5}$$

so that (cf. Eqs. (1.1) and (1.2))

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$$V_{\frac{1}{2},-\frac{1}{2}}(x,y) = K(x,y), \tag{1.6}$$
$$V_{\frac{1}{2},\frac{1}{2}}(x,y) = L(x,y). \tag{1.7}$$

We note the following explicit representations for the generalized Voigt function $V_{\mu,\nu}(x,y)$ [11, p. 113, Eqs. (10) and (11)]:

$$V_{\mu,\nu}(x,y) = 2^{\mu-\frac{1}{2}}x^{\nu+\frac{1}{2}}\sum_{m,n=0}^{\infty}\frac{(-x^2)^m(-2y)^n}{m!n!\Gamma(\nu+m+1)}\Gamma\left(\frac{1}{2}(\mu+\nu+2m+n+1)\right) \tag{1.8}$$

and

$$\begin{aligned} V_{\mu,\nu}(x,y) = \frac{2^{\mu-\frac{1}{2}}x^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)}\Bigg\{&\Gamma\left(\frac{1}{2}(\mu+\nu+1)\right)\times\Psi_2\left[\frac{1}{2}(\mu+\nu+1);\nu+1,\frac{1}{2};-x^2,y^2\right] \\ &-2y\Gamma\left(\frac{1}{2}(\mu+\nu+2)\right)\Psi_2\left[\frac{1}{2}(\mu+\nu+2);\nu+1,\frac{3}{2};-x^2,y^2\right]\Bigg\} \quad (Re(\mu+\nu)>-1), \end{aligned} \tag{1.9}$$

where Ψ_2 denotes one of Humbert's confluent hypergeometric functions of two variables, defined by (see, e.g. [13, p. 26 (22)]):

$$\Psi_2[a;b,c;x,y] = \sum_{m,n=0}^{\infty}\frac{(a)_{m+n}x^my^n}{(b)_m(c)_nm!n!} \quad (\max(|x|,|y|)<\infty). \tag{1.10}$$

Since the generalized Voigt function $V_{\mu,\nu}(x,y)$ can be expressed in terms of integral representation involving the Bessel function $J_\nu(x)$ (cf. Eq. (1.5)), the properties of this function assume noticeable importance. Indeed, each of these properties will naturally lead to various other needed properties for the generalized Voigt function $V_{\mu,\nu}(x,y)$. Motivated by the important role of the Voigt functions in several diverse fields of physics and the contributions in [5,6,11,12] toward the unification and generalization of the Voigt functions, this work aims at investigating several properties and representations for the generalized Voigt function $V_{\mu,\nu}(x,y)$. We establish some differential and pure recurrence relations, connections with certain hypergeometric functions and polynomials, expansions and integral representations for the generalized Voigt function $V_{\mu,\nu}(x,y)$. Also, we discuss the link for the various results, which are presented in this paper, with known results.

2. Recurrence relations

Denote $V_{\mu,\nu}(x,y)$ by $V_{\mu,\nu}$, then based on (1.8), we obtain the following differential relations:

$$\frac{\partial}{\partial x}\{V_{\mu,\nu}\} = \left(\frac{\nu+\frac{1}{2}}{x}\right)V_{\mu,\nu} - V_{\mu+1,\nu+1}, \tag{2.1}$$
$$\frac{\partial}{\partial y}\{V_{\mu,\nu}\} = -V_{\mu+1,\nu}. \tag{2.2}$$

Further, one has

$$\frac{\partial^2}{\partial x\partial y}\{V_{\mu,\nu}\} = \frac{\partial^2}{\partial y\partial x}\{V_{\mu,\nu}\} = \left(\frac{\nu+\frac{1}{2}}{x}\right)\frac{\partial}{\partial y}V_{\mu,\nu} - V_{\mu+2,\nu+1}. \tag{2.3}$$

In view of the definition of the digamma function ψ [1, p. 90 (2.54)] the derivative of the generalized Voigt function $V_{\mu,\nu}(x,y)$ (cf. (1.8)) with respect to its orders ν and μ lead to

$$\begin{aligned} \frac{\partial}{\partial\nu}\{V_{\mu,\nu}\} = \ln(x)V_{\mu,\nu} + &\left\{2^{\mu-\frac{1}{2}}x^{\nu+\frac{1}{2}}\times\sum_{m,n=0}^{\infty}\frac{(-x^2)^m(-y)^n\Gamma\frac{1}{2}(\mu+\nu+2m+n+1)}{m!n!\Gamma(\nu+m+1)}\right\} \\ &\times\left\{\psi\left[\frac{1}{2}(\mu+\nu+2m+n+1)\right]-\psi[(\nu+m+1)]\right\}, \end{aligned} \tag{2.4}$$

and

$$\frac{\partial}{\partial\mu}\{V_{\mu,\nu}\} = \ln(2)V_{\mu,\nu} + \left\{2^{\mu-\frac{1}{2}}x^{\nu+\frac{1}{2}}\times\sum_{m,n=0}^{\infty}\frac{\left(\frac{-x2}{4z}\right)^m\left(\frac{-y}{\sqrt{z}}\right)^n\Gamma\frac{1}{2}(\mu+\nu+2m+n+1)}{m!n!\Gamma(\nu+m+1)}\right\}\times\psi\left[\frac{1}{2}(\mu+\nu+2m+n+1)\right], \tag{2.5}$$

respectively.

Suppose we multiply (1.8) by x^ν and then differentiate with respect to x.

This gives us

$$\frac{\partial}{\partial x}\{x^\nu V_{\mu,\nu}\} = x^\nu V_{\mu+1,\nu-1} + \frac{x^{\nu-1}}{2}V_{\mu,\nu}. \tag{2.6}$$