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Wendel's and Gautschi's inequalities: Refinements, extensions, and a class of logarithmically completely monotonic functions

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ABSTRACT

In the article, sufficient and necessary conditions that a class of functions involving ratio of Euler's gamma functions and originating from Wendel's and Gautschi's inequalities are logarithmically completely monotonic are presented. From this, Wendel's, Gautschi's, Kershaw's, Laforgia's, Bustoz-Ismail's, Merkle's and Elezović-Giordano-Pečarić's inequalities are refined, extended and sharpened, and a double inequality on the divided differences of the psi and polygamma functions is deduced straightforwardly.

1. Introduction

Recall [4,10,24,34,43] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geqslant 0, \tag{1}$$

for $x \in I$ and $n \geqslant 0$. Recall also [2,31,34–36] that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geqslant 0, \tag{2}$$

for all $k \in \mathbb{N}$ on I. It has been presented explicitly in [4,31,34,39] that a logarithmically completely monotonic function must be completely monotonic, but not conversely. In [4, Theorem 1.1] and [12] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [13, Theorem 4.4]. In recent years, the notion "logarithmically completely monotonic function" has been adopted in many articles such as [4,7–9,12,16,18,19,24,28,30,32,33,35,36,40–42,45] and the references therein.

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In order to establish the classical asymptotic relation

$$\lim_{x\to\infty}\frac{\Gamma(x+a)}{x^a\Gamma(x)}=1, \tag{3}$$

for real a and x, using Hölder's integral inequality, the following double inequality was proved in [44]:

$$\left(\frac{x}{x+a}\right)^{1-a}\leqslant\frac{\Gamma(x+a)}{x^a\Gamma(x)}\leqslant 1 \tag{4}$$

for $0<a<1$ and $x>0$, where $\Gamma(x)$ denotes the well known classical Euler's gamma function Γ defined for $x>0$ as

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,dt. \tag{5}$$

This inequality can be rewritten for $0<a<1$ and $x>0$ as

$$(x+a)^{1-a}\geqslant\frac{\Gamma(x+1)}{\Gamma(x+a)}\geqslant x^{1-a}. \tag{6}$$

In [11], along with another line, the following two double inequalities were established for $n\in\mathbb{N}$ and $0\leqslant s\leqslant 1$:

$$\exp[(1-s)\psi(n+1)]\geqslant\frac{\Gamma(n+1)}{\Gamma(n+s)}\geqslant n^{1-s} \tag{7}$$

and

$$(n+1)^{1-s}\geqslant\frac{\Gamma(n+1)}{\Gamma(n+s)}\geqslant n^{1-s}. \tag{8}$$

It is clear that the upper bound in inequality (8) is not better and the range in inequality (8) is not larger than the corresponding ones in (4) or (6).

Motivated by the paper [11], among other things, the following double inequality was showed for $0<s<1$ and $x\geqslant 1$ in [14]:

$$\left(x+\frac{s}{2}\right)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]^{1-s}. \tag{9}$$

It is easy to see that inequality (9) refines inequalities (4), (6), (8) and the left hand side inequality in (7).

In [15], a method of obtaining inequalities of the type:

$$(k+\alpha)^{\lambda-1}<\frac{\Gamma(k+\lambda)}{\Gamma(k+1)}<(k+\beta)^{\lambda-1}, \tag{10}$$

for $\lambda>0$ and $k\geqslant 0$ was presented, where α and β are independent of k.

Inequalities (9) and (10) have been investigated along with two directions.

A standard argument shows that inequality (9) can be rearranged as

$$\frac{s}{2}<\left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(1-s)}-x<\sqrt{s+\frac{1}{4}}-\frac{1}{2}. \tag{11}$$

Therefore, the first direction is to consider the monotonicity of the general function

$$z_{s,t}(x)=\begin{cases}\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}-x, & s\neq t\\ e^{\psi(x+s)}-x, & s=t\end{cases} \tag{12}$$

in $x\in(-\alpha,\infty)$, where s and t are two real numbers and $\alpha=\min\{s,t\}$. In [6,10,20,21,27,37], it was obtained that the function $z_{s,t}(x)$ is either convex and decreasing for $|t-s|<1$ or concave and increasing for $|t-s|>1$.

The second direction is to consider the monotonicity, complete monotonicity or logarithmically complete monotonicity of the function

$$H_{a,b,c}(x)=(x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{13}$$

for $x\in(-\rho,\infty)$, where a, b and c are real numbers and $\rho=\min\{a,b,c\}$. It is clear that

$$\frac{1}{H_{a,b,c}(x)}=H_{b,a,c}(x). \tag{14}$$

In [5, Theorems 1 and 3] it was revealed for $a+1\geqslant b>a$ that $H_{b,a,c}(x)$ is completely monotonic in $(\max\{-a,-c\},\infty)$ if $c\leqslant\frac{a+b-1}{2}$ and that $H_{a,b,c}(x)$ is completely monotonic in $(\max\{-b,-c\},\infty)$ if $c\geqslant a$. In [5, Theorem 7] it was demonstrated that $H_{1,s,s/2}(x)$