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Exact solutions to a nonlinearly dispersive Schrödinger equation

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Abstract

By using the theory of dynamical systems to a nonlinearly dispersive Schrödinger equation, the existence of solitary patterns, compactons, smooth and non-smooth periodic patterns and breather solutions is obtained. Under different parametric conditions, various sufficient conditions to guarantee existence of the above solutions are given. In some simple conditions, exact explicit and implicit parametric representations of solutions are given.

Keywords: Solitary patterns; Periodic patterns; Compactons; Breather solutions; Theory of bifurcation; Nonlinearly dispersive Schrödinger equation

1. Introduction

It is well known that the exact solutions of the nonlinearly Schrödinger type equations have been extensively studied in the field of theoretical physics (see [1–3] and cited reference therein).

In this paper, we consider the following nonlinearly dispersive Schrödinger equation (NLS(m, n)equation):

$$\mathrm{i}u_t + (u|u|^{n-1})_{xx} + \mu u|u|^{m-1} = 0, \tag{1.1}$$

where n is a positive integer, m is a integer, $\mu = \pm 1$ and $\mathrm{i}^2 = -1$. When $m = 3$, $n = 1$, (1.1) becomes the usual nonlinear Schrödinger (NLS) equation:

$$\mathrm{i}u_t + u_{xx} + \mu u|u|^2 = 0. \tag{1.2}$$

We shall use the method of dynamical systems (see [4–9]) to find the exact solutions of (1.1). By considering the dynamics of the solutions determined by the nonlinear wave systems, we shall generally give all possible

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explicit exact solutions for (1.1) in the different parameter regions by using the elliptic functions and hyperbolic functions (see [10]).

To find exact solutions for (1.1), we take the following transformation: $u(x, t) = U(x)e^{i\sigma t}$. Then, (1.1) can be become the ordinary differential equation:

$$-\sigma U + n(n-1)U^{n-2}U'^2 + nU^{n-1}U'' + \mu U^m = 0, \tag{1.3}$$

where "′" is the derivative with respect to x.

(1.3) is equivalent to the following system:

$$\frac{dU}{dx} = y, \quad \frac{dy}{dx} = \frac{1}{nU^{n-2}}(\sigma - \mu U^{m-1}) - (n-1)\frac{1}{U}y^2, \tag{1.4}$$

which has the first integral

$$H(U, y) = U^{2(n-1)}y^2 - \frac{1}{n}\left(\frac{2\sigma}{n+1}U^{n+1} - \frac{2\mu}{m+n}U^{m+n}\right). \tag{1.5}$$

When $\mu = 1$, (1.1) is referred to as the focusing (+) branch and is signed as $\mathrm{NLS}^+(m, n)$ equation

$$iu_t + (u|u|^{n-1})_{xx} + u|u|^{m-1} = 0. \tag{1.6}$$

When $\mu = -1$, (1.1) is referred to as the focusing (−) branch and is signed as $\mathrm{NLS}^-(m, n)$ equation:

$$iu_t + (u|u|^{n-1})_{xx} - u|u|^{m-1} = 0. \tag{1.7}$$

More recently, Yan Zhenya (see [11]) considered the envelope compactons and solitary patterns of Eq. (1.1). He stated in [11] that "when $n = 1$ and $m < 1$, $\mathrm{NLS}^+(m, 1)$ equation is shown to possess envelope compactons" and "when $n > 1$ $\mathrm{NLS}^+(n, n)$ equation has envelope compactons solution". But in this paper, we shall show that $\mathrm{NLS}^+(n, n)$ equation has compacton solution only when $n = 2$ and $\mathrm{NLS}^+(m, 1)$ equation has no compacton solution.

The rest of this paper is organized as follows. In Section 2, we discuss the bifurcations of phase portraits of (1.6). Corresponding to all bounded orbits, we give all possible exact explicit parametric representations of the solutions for Eq. (1.6). In Section 3, we discuss the bifurcations of phase portraits of (1.7). Corresponding to all bounded orbits, we give all possible exact explicit parametric representations of the solutions for Eq. (1.7).

2. The exact solutions of $\mathrm{NLS}^+(m, n)$ equation

When $\mu = 1$, (1.4) and (1.5) become

$$\frac{dU}{dx} = y, \quad \frac{dy}{dx} = \frac{1}{nU^{n-2}}(\sigma - U^{m-1}) - (n-1)\frac{1}{U}y^2 \tag{2.1}$$

and

$$H(U, y) = U^{2(n-1)}y^2 - \frac{1}{n}\left(\frac{2\sigma}{n+1}U^{n+1} - \frac{2}{m+n}U^{m+n}\right). \tag{2.2}$$

Let $M(u_e, 0)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point $(u_e, 0)$ and $J(u_e, 0)$ be its Jacobin determinant.

By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if $J > 0$ and $\mathrm{Trace}(M(u_e, 0)) = 0$ then it is a center point; if $J > 0$ and $(\mathrm{Trace}(M(u_e, 0)))^2 - 4J(u_e, 0) > 0$ then it is a node; if $J = 0$ and the Poincare index of the equilibrium point is 0 then it is a cusp. By using the above fact, we have the following results.

(I) When $n = 1$, (2.1) becomes

$$\frac{dU}{dx} = y, \quad \frac{dy}{dx} = \sigma U - U^m, \tag{2.3}$$