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Solution of a class of Volterra integral equations with singular and weakly singular kernels

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Abstract

A class of Volterra integral equations are solved in this paper, where the involved kernel may be weakly singular, singular, and hypersingular. For various cases, existence and uniqueness of such Volterra integral equations are established. Furthermore, by reducing such equations to ordinary differential equations, analytic solutions have been determined explicitly. When restricting the solutions to certain special function classes, the number of suitable solutions is given.

Keywords: Volterra integral equation; Singular kernel; Ordinary differential equation; Analytic solution

1. Introduction

The purpose of this paper is to derive the solution to a linear Volterra integral equation of the form

$$y(t) = g(t) + \int_0^t \frac{s^{\mu-\nu}}{t^{\mu}} y(s)\,\mathrm{d}s, \quad 0 < t < T, \tag{1}$$

where μ and ν are constants, $g(t)$ is a known function, and $y(t)$ is an unknown function to be determined. Such a class of Volterra integral equations are encountered in some applications. For instance, the case of $\mu > 0$ and $\nu = 1$ arises in heat conduction problems posed by mixed boundary conditions [1,2]. This situation has been intensively studied through analytical and numerical approaches by many researchers [3–5]. Because of the special form of the kernel $k(t,s) = s^{\mu-\nu}/t^{\mu}$, some usual conclusions including existence and uniqueness of Volterra integral equations [6] cannot be directly applied since the kernel involved at the origin is singular for $\mu > 0$ with respect to t or $\mu < \nu$ with respect to s, even hypersingular for $\mu > 1$ or $\nu - \mu > 1$.

In particular, when $\nu = 1$, for the case of $0 < \mu < 1$, the kernel is weakly singular at $t = 0$, and so Eq. (1) is a weakly singular Volterra integral equation. To date, considerable studies have been made for weakly singular

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Volterra integral equations, in particular for those with Abel-type kernel [7–11]. However, for Volterra integral Eq. (1) with weakly singular kernel, related works are limited. It has been shown in [12] that Eq. (1) admits a unique solution in the continuity class $C^m[0,T]$ if $g \in C^m[0,T]$ and $\mu > 1$. Nevertheless, for $0 < \mu \leqslant 1$ together with $v = 1$, Eq. (1) possesses an infinite number of solutions in $C[0,T]$, of which only one lies in $C^1[0,T]$ [3]. Such conclusions are given in suitable function spaces with specified norms. Recently, some schemes have been proposed to determine the numerical solution of Eq. (1) with $v = 1$, $\mu > 1$ and $v = 1$, $0 < \mu \leqslant 1$ [4,5,13,14]. In effect, for two negative constants μ and v, Eq. (1) is also singular or hypersingular depending on $v - \mu = 1$ and $v - \mu > 1$, respectively. So far, for such cases, existence and uniqueness, and the number of the solutions in certain function classes have not been analyzed. A study on Volterra integral Eq. (1) with arbitrary two constants μ and v is most desirable.

In this paper, we consider a more general case, i.e. μ and v are arbitrary two constants. By transforming the above Volterra integral equation in question into an ordinary differential equation, we derive analytically a general solution to Eq. (1), which has two advantages, one being that the obtained solution is given in explicit form, and the other being that it can serve as a benchmark of examining the accuracy of numerical solutions. Furthermore, when restricting the obtained solution to certain function classes, some new results on the number of the suitable solutions can be reduced. Some previous results can be taken as special cases of the present results.

2. Method of solution

Now we turn our attention to solving Volterra integral Eq. (1). An initial inspection of Eq. (1) indicates that the value of the unknown function $y(t)$ at $t = 0$ is defined or undefined depending the behavior of the desired solution $y(t)$ near $t = 0$. For example, for $v = 1$, $\mu \neq 1$

$$\lim_{t\to 0^+} \int_0^t \frac{s^{\mu-1}}{t^\mu} y(s)\,\mathrm{d}s = \frac{y(0)}{\mu - 1}, \tag{2}$$

and

$$y(0) = \frac{\mu}{\mu - 1} g(0), \tag{3}$$

inferring that in this case $t = 0$ is a removable singularity point. Consequently, for convenience in what follows we first confine ourselves to the region of $t > 0$.

To derive a general solution of Volterra integral Eq. (1), we introduce a new unknown function such that

$$\varphi(t) = \int_0^t s^{\mu-v} y(s)\,\mathrm{d}s, \quad 0 < t < T. \tag{4}$$

Lemma 1. *Let $\alpha < \mu - v + 1$, $M > 0$. Suppose that $y \in C(0,T]$, and $|y(t)| \leqslant Mt^{-\alpha}$ as $t \to 0^+$. Then*

$$\lim_{t\to 0^+} \varphi(t) = 0. \tag{5}$$

Proof. Since there exists a vicinity $(0,\varepsilon)(\varepsilon > 0)$ such that $|y(t)| \leqslant Mt^{-\alpha}$, then

$$|\varphi(t)| \leqslant \int_0^t |s^{\mu-v} y(s)|\,\mathrm{d}s \leqslant M \int_0^t s^{\mu-v-\alpha}\,\mathrm{d}s, \quad t \in (0,\varepsilon). \tag{6}$$

So $\lim_{t\to 0^+} \varphi(t) = 0$ follows $\alpha < \mu\text{-}v\text{+}1$. □

If $y(t)$ is of form $Mt^{-\alpha}$, we further can readily prove the following Lemma 2.

Lemma 2. *Suppose that $y \in C(0,T]$, and $y(t) = Mt^{-\alpha}$ $(M > 0)$ as $t \to 0^+$. Then the integral in (4) is convergent, and moreover $\lim_{t\to 0^+} \varphi(t) = 0$ for $\alpha < \mu - v + 1$, and otherwise the integral is divergent for $\alpha \geqslant \mu - v + 1$.*

In the following, we only consider the case of a convergent integral in (4) or further $\lim_{t\to 0^+} \varphi(t) = 0$. Moreover, the following lemma holds.