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Variants of algebraic wavelet-based multigrid methods: Application to shifted linear systems

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Abstract

In this paper, we describe some new variants and applications of the wavelet algebraic multigrid method. This method combines the algebraic multigrid method (a well known family of multilevel techniques for solving linear systems, without use of knowledge of the underlying problem) and the discrete wavelet transform. These two techniques can be combined in several ways, obtaining different methods for solution of linear systems; these can be used alone or as preconditioners for Krylov iterative methods.

These methods can be applied for solution of linear systems with shifted matrices of the form $A - hI$, whose efficient solution is very important for implicit ODE methods, unsteady PDEs, computation of eigenvalues of large sparse matrices and other important problems.

Keywords: Algebraic multigrid; Wavelets; Linear systems; Sparse matrices; Shifted matrices; Preconditioners

1. Introduction

The discrete wavelet transform (DWT) is a numerical tool of increasing importance in many fields, whose possibilities of application are still being explored. The main characteristic of this tool is its ability to compress or to analyze data. Its properties have been applied successfully in image processing, signal processing, and are applied as well in other fields-like numerical linear algebra. The DWT has already proved its usefulness in some problems, such as solution of boundary integral equations, where the DWT is used to "compress" the matrix, turning a dense linear system into a sparse one [1].

The core algorithm for the DWT, the pyramidal algorithm, has a multilevel structure which reminds of the multigrid method for solution of linear systems. Therefore, it is quite natural that many papers have been devoted to explore the connection between these two methods, proposing a wide variety of wavelet–multigrid methods.

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Most of these papers describe wavelet-based, multigrid-like algorithms, using the wavelets as basis for discretization for solution of partial differential equations (PDEs). These methods can be very efficient, but they are not general, since a full knowledge of the discretization and underlying problem is needed to apply them. On the other hand, there are other multigrid-like algorithms, which can be applied to any linear system, no matter where it comes from. Only the matrix and the right hand side needs to be known. These multigrid methods are called "algebraic".

The present paper is concerned with the wavelet algebraic multigrid methods (WAMG), that is, methods combining multigrid and wavelets, and that do not need any knowledge of the underlying problem; only the coefficient matrix and the right hand side are needed.

As far as we know, there are only a few papers devoted to WAMG methods. The (in our view) main algorithm was proposed first by Wang et al. in [2] and revisited by Pereira et al. in [3]; another important algorithm was proposed as well by De Leon in [4] (we will discuss these algorithms in depth in chapter 4). However, there are several interesting variants of these algorithms that were not discussed in these papers. Here we will propose two variants of WAMG algorithms. The first is based on using the decomposition of the linear system induced by the DWT. The second is a technique already known in the multigrid method, where the cost of the multigrid cycles is reduced by skipping operations in some levels or grids.

An important observation (which will be discussed at the end of the paper) about these algebraic wavelet–multigrid algorithms is that they are specially suited for problems where many linear systems with shifted matrices $A - hI$ must be solved. This is very relevant for solution of time-dependent partial differential equations (PDEs) and systems of ordinary differential equations.

The paper is organized as follows: it starts by giving a brief outline of both methods, DWT and multigrid. Then, the wavelet algebraic algorithms proposed in [2–4] shall be discussed. After that, we will comment our ideas about these algorithms, and about generation of new multigrid-like algorithms. The numerical performance of algebraic wavelet-based multigrid algorithms as preconditioner shall then be compared experimentally with some of the most popular preconditioners, the different versions of the incomplete LU decomposition (ILU0 and ILUT) [5]. Finally, we will discuss the application of these algorithms for solution of shifted linear systems.

2. Multigrid

The multigrid method was devised to solve efficiently linear systems

$$Tx = b \tag{1}$$

arising from discretization of partial differential equations. The name "multigrid" comes form the fact that, the original problem was discretized in several, progressively coarser grids, which were used to build iteratively the solution. To describe the multigrid method, it is enough to consider only two grids, the original "fine" one, and a "coarser" one. Once discretized the original problem in these two grids, the following observations give the main idea:

- Simple iterative methods such as Gauss–Seidel or Jacobi (which, as usual, shall be called smoothers or relaxation methods) are quite effective in removing the high frequency components of the error, while are slow in removing low frequency components of the error.
- The low frequency components of the error might become high frequency components (and, therefore, easy to wipe out) if same the problem could be interpolated into a coarser grid.

These two observations led to a first two-grid algorithm:

Algorithm 1. Two-grid algorithm

(1) Apply a few smoothings to the finer grid system, to obtain an initial approximate solution x^{f}.
(2) Compute the residual for this approximation. $r^{\mathrm{f}} = b^{\mathrm{f}} - T^{\mathrm{f}}x^{\mathrm{f}}$. (Giving as well the residual equation, $T^{\mathrm{f}}e^{\mathrm{f}} = r^{\mathrm{f}}$.)