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Generalized Vandermonde determinants and mean values

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Abstract

In this article, the generalized Vandermonde determinants of order $n+1$ are introduced and studied systematically, a type of mean values of several positive numbers are defined by using the same type of determinants, and some of their basic properties and applications are given.

Keywords: Vandermonde determinants; Mean values; Integral identity; Elementary symmetric function; Monotonicity; Integral inequality

1. Introduction

Throughout the present investigation, we assume that

$$\boldsymbol{x} = (x_0, x_1, x_2, \ldots, x_n), \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\} \quad (\mathbb{N} := \{1, 2, 3, \ldots\}),$$

and

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{R}_+ = [0, \infty), \quad \text{and} \quad \mathbb{R}_+^n = [0, \infty)^n.$$

The following determinant of the Vandermonde matrix of order $n+1$ is well known:

$$V(\boldsymbol{x}) := \begin{vmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ 1 & x_2 & x_2^2 & \ldots & x_2^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{vmatrix} = \prod_{0 \leqq i < j \leqq n} (x_j - x_i). \tag{1.1}$$

Obviously, if we set

$$x_0 < x_1 < \cdots < x_n,$$

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then

$$V(\boldsymbol{x}) > 0.$$

Recently, Xiao and Zhang [3] gave the following general form of the Vandermonde determinant of order $n+1$:

$$V(\boldsymbol{x};r) := \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & x_0^{n+r} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^{n+r} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & x_2^{n+r} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^{n+r} \end{vmatrix}, \tag{1.2}$$

where

$$x_i \in \mathbb{R}_+ \quad (0 \leqq i \leqq n) \quad \text{and} \quad r \in \mathbb{R}.$$

In fact, Xiao and Zhang [3] also proved an integral identity relating $V(\boldsymbol{x};r)$ with $V(\boldsymbol{x})$ as follows:

$$V(\boldsymbol{x};r) = \prod_{k=1}^{n}(k+r)\cdot V(\boldsymbol{x})\cdot \int_E \left(\sum_{i=0}^{n} x_i t_i\right)^r \mathrm{d}t_1\,\mathrm{d}t_2\cdots \mathrm{d}t_n, \tag{1.3}$$

where

$$t_0 = 1 - \sum_{i=1}^{n} t_i$$

and

$$E := \left\{(t_1,t_2,\ldots,t_n) : \sum_{i=1}^{n} t_i \leqq 1 \quad (t_i \in \mathbb{R}_+;\ i=1,\ldots,n)\right\}.$$

Subsequently, Xiao et al. [4] studied the extended mean values $E(p,q;\boldsymbol{x})$ by using the generalized Vandermonde determinant $V(\boldsymbol{x};r)$ in (1.2) above and established the following result.

Theorem 1. *Let*

$$\boldsymbol{x} = (x_0,x_1,\ldots,x_n) \in \mathbb{R}_+^{n+1} \quad (x_i \neq x_j \text{ when } i \neq j).$$

Then the extended mean values $E(p,q;\boldsymbol{x})$ of the $(n+1)$-tuple $\boldsymbol{x}$ with two parameters p and q are increasing strictly with respect to both $p \in \mathbb{R}$ and $q \in \mathbb{R}$, where

$$E(p,q;\boldsymbol{x}) := \begin{cases} \left(\prod_{i=1}^{n} \frac{i+q}{i+p}\cdot \frac{V(\boldsymbol{x};p,0)}{V(\boldsymbol{x};q,0)}\right)^{1/(p-q)} & \left((p-q)\prod_{i=1}^{n}[(i+p)(i+q)] \neq 0\right), \\ \left(\frac{(-1)^{q+1}(-q-1)!(q+n)!}{\prod_{i=1}^{n}(i+p)}\cdot \frac{V(\boldsymbol{x};p,0)}{V(\boldsymbol{x};q,1)}\right)^{1/(p-q)} & (p \neq q;\ q \in \{-1,-2,\ldots,-n\}), \\ \left(\frac{(-1)^{q-p}(-q-1)!(q+n)!}{(-p-1)!(p+n)!}\cdot \frac{V(\boldsymbol{x};p,1)}{V(\boldsymbol{x};q,1)}\right)^{1/(p-q)} & (p \neq q;\ p,q \in \{-1,-2,\ldots,-n\}), \\ \exp\left(\frac{V(\boldsymbol{x};p,1)}{V(\boldsymbol{x};p,0)} - \sum_{i=1}^{n}\frac{1}{i+p}\right) & (p = q \notin \{-1,-2,\ldots,-n\}), \\ \exp\left(\frac{V(\boldsymbol{x};p,2)}{2V(\boldsymbol{x};p,1)} - \sum_{\substack{i=1 \\ (i\neq -p)}}^{n}\frac{1}{i+p}\right) & (p = q;\ q \in \{-1,-2,\ldots,-n\}) \end{cases} \tag{1.4}$$