Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 202 (2008) 348–359
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Some phenomenon of the powers of certain tridiagonal and asymmetric matrices

Roman Wituła, Damian Słota *

Institute of Mathematics, Silesian University of Technology, Kaszubska 23, Gliwice 44-100, Poland

Abstract

In this paper, a new method of calculation of real (and even complex) powers of some asymmetric matrices obeying constance tridiagonals is presented.

Keywords: Powers of matrices; Asymmetric matrices; Tridiagonal matrices

1. Introduction

Let us set

$$\mathbf{T}_n(a,b,c) := \begin{bmatrix} a & c & & & & \\ b & a & c & & 0 & \\ & b & a & c & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & b & a & c \\ & & & & b & a \end{bmatrix}_{n\times n}, \tag{1}$$

where $a,b,c \in \mathbb{C}$, $n \in \mathbb{N}$.

In the course of preparing paper [1] focused on a certain property of tridiagonal matrices and general constant-diagonals matrices, the authors came across a certain phenomenon connected with the real powers of tridiagonal matrix $\mathbf{T}_n(x,1,1)$ and then with the complex powers of matrices of a more general form:

$$\mathbf{F}_3(a,b,c,A) := \begin{bmatrix} a & b & c \\ b & A & b \\ c & b & a \end{bmatrix}, \tag{2}$$

* Corresponding author.
E-mail address: d.slota@polsl.pl (D. Słota).

0096-3003/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2008.02.023

where $a, b, c \in \mathbb{C}$ and $A := a + c$. This phenomenon consists of finding Binet's formulas of elements of positive integers powers of given matrix and next to extend these ones to complex powers. Our considerations are extended also to matrices of asymmetric forms:

$$\mathbf{G}_3(a,b,c,A,y) := \begin{bmatrix} a & yb & y^2c \\ b & A & yb \\ c & b & a \end{bmatrix} \tag{3}$$

and

$$\mathbf{G}_4(a,b,c,d,e,A,y) := \begin{bmatrix} a & yb & y^2c & y^3e \\ b & A & yd & y^2c \\ c & d & A & yb \\ e & c & b & a \end{bmatrix} \tag{4}$$

for any $a, b, c, e, y \in \mathbb{C}$, $A := a + yc$, $d := b + ye$.

2. Positive integers powers of $\mathbf{F}_3$

We shall start our deliberations with a simple case, that is with describing the positive integers powers of matrix (2).

Lemma 2.1. *Let $a_1, b_1, c_1 \in \mathbb{C}$ and $A_1 := a_1 + c_1$. Then we have*:

$$(\mathbf{F}_3(a_1,b_1,c_1,A_1))^k = \mathbf{F}_3(a_k,b_k,c_k,A_k) \tag{5}$$

for every $k \in \mathbb{N}$, where the following formulas hold:

$$A_k = a_k + c_k, \tag{6}$$
$$a_{k+1} = a_1a_k + b_1b_k + c_1c_k, \tag{7}$$
$$b_{k+1} = A_1b_k + A_kb_1, \tag{8}$$
$$c_{k+1} = a_1c_k + b_1b_k + c_1a_k, \tag{9}$$
$$A_{k+1} = 2b_1b_k + A_1A_k. \tag{10}$$

The proof of Lemma by induction follows.

Corollary 2.2. *We have*

$$a_k = c_k + (a_1 - c_1)^k. \tag{11}$$

Proof. We note that (7)–(9) $\Rightarrow a_{k+1} - c_{k+1} = (a_1 - c_1)(a_k - c_k)$, $k = 1,2,\ldots$ □

Corollary 2.3. *Now let us set* ($x, y \in \mathbb{C}$):

$$z := a_1 - c_1, \quad A_1 := \frac{1}{2}(x+y), \quad b_1 := \frac{\sqrt{2}}{4}(x-y).$$

Then the following identities (the Binet's formulas for a_k, b_k, c_k and A_k, see for example [2–4]) can be easily generated:

$$a_k(x,y,z) = \frac{1}{4}(x^k + y^k + 2z^k), \tag{12}$$

$$c_k(x,y,z) = \frac{1}{4}(x^k + y^k - 2z^k), \tag{13}$$

$$b_k(x,y) = \frac{\sqrt{2}}{4}(x^k - y^k) \tag{14}$$