Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 195 (2008) 309–315
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Monte Carlo method for solving Fredholm integral equations of the second kind

R. Farnoosh [a,*], M. Ebrahimi [b]

[a] Faculty of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran
[b] Department of Mathematics, Slamic Azad University of Karaj Branch, Karaj, Iran

Abstract

In this paper, we present a numerical method based on random sampling for the solution of Fredholm integral equations of the second kind. This method is a Monte Carlo method based on the simulation of a continuous Markov chain. To illustrate the usefulness of this technique we apply it to some test problems. Numerical results are performed in order to show the efficiency and accuracy of the present work.

Keywords: Monte Carlo method; Markov chain; Simulation; Fredholm integral equation

1. Introduction

It is well known that Monte Carlo methods are preferable for solving large sparse systems of linear algebraic equations, such as those arising from approximations of partial differential equations [1,2]. Such methods are good for solving integral equations. One of the earliest methods for solving integral equations by a Monte Carlo method was proposed by Albert [3] and was later developed in [4,5]. In a number of papers Monte Carlo methods for the computation of integrals depending on a parameter, integral operators and the solution of integral equations were proposed and studied [6]. Monte Carlo methods are a classical tool for solving high dimensional integral equations. Basic applications include neutron transport and thermal radiation can be found in [7–9], respectively.

In this paper, we will consider the following Fredholm integral equation of the second kind:

$$u(x) = f(x) + \lambda \int_0^1 k(x,t)u(t)\,\mathrm{d}t, \quad \lambda = 1, \quad 0 \leqslant x \leqslant 1, \tag{1}$$

where the function $f(x) \in L_2[0,1]$, the kernel $k(x,t) \in L_2([0,1]\times[0,1])$ are given and $u(x) \in L_2[0,1]$ is the unknown function to be determined.

* Corresponding author.
E-mail addresses: rfarnoosh@iust.ac.ir (R. Farnoosh), mo_ebrahimi@kiau.ac.ir (M. Ebrahimi).

0096-3003/$ - see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2007.04.097

Numerical methods including quadrature, collocation, Galerkin, Petrov Galerkin and Wavelet Petrov Galerkin methods for Eq. (1) has been studied in [10,11]. Here we want to propose a numerical method based on random sampling to find an approximation to the unknown function $u(x)$.

Associated with integral Eq. (1) two subproblems can be distinguished. In the first case, we seek to approximate the full solution function $u(x)$ in some way (e.g., by solving on a grid and interpolating or by finite difference approximation etc.). We call this the problem of global solution. In the second case, we want to approximate the value $u(x_0)$ of the solution in a single point x_0 or the value of a functional of $u(x)$, e.g., the integral over $[0,1]$. This is called the local solution problem [12].

While deterministic numerical methods such as Nyström, collocation, FDM usually aim at solving the global problem, the classical Monte Carlo approach is directed to the local solution. Monte Carlo methods are well understood in this situation and are generally acknowledged to bring advantages (at least for high dimensional problems) over the deterministic approaches.

The Monte Carlo complexity, i.e., the complexity of the stochastic solution of this problem, is analyzed in [12]. The results show that even in the global case Monte Carlo algorithms can perform better than deterministic ones, although the difference is not as large as in the local case.

In the present work, we show how Monte Carlo method can be used to solve linear Fredholm integral equations. Our idea for solving integral Eq. (1) by a Monte Carlo method is use continuous Markov chain with state space $[0,1]$, for simulation.

2. Overview of the method

Eq. (1) may be write in the operational form as

$$u(x) = f(x) + (Ku)(x), \tag{2}$$

or

$$u = f + Ku, \tag{3}$$

where K is an integral operator for the integral in Eq. (1) which maps the function $u(x)$, as an input, into an output

$$(Ku)(x) = \int_0^1 k(x,t)u(t)\,\mathrm{d}t. \tag{4}$$

$(Ku)(x)$ is called the first iteration of u with respect to the kernel k. The second iteration is

$$K[(Ku)](x) = (K^2u)(x) = \int_0^1 \int_0^1 k(x,t)k(t,t_1)u(t_1)\,\mathrm{d}t\,\mathrm{d}t_1. \tag{5}$$

Proceeding recursively we obtain the nth iteration of u with respect to the kernel k as

$$K[(K^{n-1}u)](x) = (K^nu)(x) = \int_0^1 k(x,t_{n-1})K^{n-1}u(t_{n-1})\,\mathrm{d}t_{n-1}. \tag{6}$$

Let us assume that

$$|K| = \underbrace{\sup}_{[0,1]} \int_0^1 |k(x,t)|\,\mathrm{d}t < 1. \tag{7}$$

Under this assumption, we can solve (1) by applying the following recursive equation:

$$u^{(n+1)} = Ku^{(n)} + f, \quad n = 1,2,\ldots. \tag{8}$$

If $u^{(0)}=0$ and $K^0 \equiv 0$ then from Eq. (8), we obtain

$$u^{(n+1)} = f + Kf + \cdots + K^nf = \sum_{m=0}^{n} K^mf. \tag{9}$$