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Global existence and blow-up for degenerate and singular parabolic system with localized sources

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Abstract

This paper deals with the blow-up properties of the solution to degenerate and singular parabolic system with localized sources and homogeneous Dirichlet boundary conditions. The existence of a unique classical nonnegative solution is established and the sufficient conditions for the solution exists globally and blows up in finite time are obtained. Furthermore, under certain conditions, it is proved that the blow-up set of the solution is the whole domain.

Keywords: Degenerate and singular parabolic system; Blow-up; Global existence; Blow-up set; Localized sources

1. Introduction

In this paper, we consider the following degenerate and singular nonlinear reaction–diffusion equations with localized sources:

$$\begin{cases} x^{q_1}u_t - (x^{r_1}u_x)_x = v^{p_1}(x_0,t), & (x,t)\in(0,a)\times(0,T),\\ x^{q_2}v_t - (x^{r_2}v_x)_x = u^{p_2}(x_0,t), & (x,t)\in(0,a)\times(0,T),\\ u(0,t)=u(a,t)=v(0,t)=v(a,t)=0, & t\in(0,T),\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in[0,a], \end{cases} \tag{1.1}$$

where $T>0$, $a>0$, $r_1,r_2\in[0,1)$, $|q_1|+r_1\neq 0$, $|q_2|+r_2\neq 0$ and $p_1>1$, $p_2>1$, $x_0\in(0,a)$ is a fixed point, and $u_0(x),\ v_0(x)\in C^{2+\alpha}(\overline{D})$ for some $\alpha\in(0,1)$ are nonnegative nontrivial functions and satisfy the compatibility condition.

Let $D=(0,a)$ and $\Omega_t=D\times(0,t)$, $\overline{D}$ and $\overline{\Omega}_t$ are their closures respectively. Since $|q_1|+r_1\neq 0$, $|q_2|+r_2\neq 0$, the coefficients of u_t, u_x, u_{xx} and v_t, v_x, v_{xx} may tend to 0 or ∞ as x tends to 0, we can regard the equations as degenerate and singular.

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Floater [1] and Chan et al. [2] investigated the blow-up properties of the following degenerate parabolic problem:

$$\begin{cases} x^q u_t - u_{xx} = u^p, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a], \end{cases} \tag{1.2}$$

where $q > 0$ and $p > 1$. Under certain conditions on the initial datum $u_0(x)$, Floater [1] proved that the solution $u(x,t)$ of (1.2) blows up at the boundary $x = 0$ for the case $1 < p \leqslant q + 1$. This contrasts with one of the results in [3], which showed for the case $q = 0$, the blow-up set of solution $u(x,t)$ of (1.2) is a proper compact subset of D. For the case $p > q + 1$, under certain conditions, in [2] Chan et al. proved that $x = 0$ is not a blow-up point and the blow-up set is a proper compact subset of D.

In [4], Chen et al. consider the following degenerate nonlinear reaction–diffusion equation with nonlocal source:

$$\begin{cases} x^q u_t - (x^\gamma u_x)_x = \int_0^a u^p \,\mathrm{d}x, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a], \end{cases}$$

they established the local existence and uniqueness of classical solution. Under appropriate hypotheses, they also got some sufficient conditions for the global existence and blow-up of positive solution. Furthermore, under certain conditions, it is proved that the blow-up set of the solution is the whole domain.

In [5], Zhou et al. discussed the following degenerate and singular nonlinear reaction–diffusion equations with nonlocal source:

$$\begin{cases} x^{q_1} u_t - (x^{r_1} u_x)_x = \int_0^a v^{p_1} \,\mathrm{d}x, & (x,t) \in (0,a) \times (0,T), \\ x^{q_2} v_t - (x^{r_2} v_x)_x = \int_0^a u^{p_2} \,\mathrm{d}x, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = v(0,t) = v(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in [0,a], \end{cases}$$

they established the local existence and uniqueness of classical solution. Under appropriate hypotheses, they obtained some sufficient conditions for the global existence and blow-up of positive solution.

In [6], Chan et al. considered the following degenerate semilinear parabolic equation:

$$\begin{cases} x^q u_t - u_{xx} = f(u(x_0,t)), & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a], \end{cases}$$

where $q \geqslant 0$, $x_0 \in (0,a)$. They established the local existence and uniqueness of classical solution and got some sufficient conditions for the blow-up of positive solution. Furthermore, under certain conditions, it is proved that the blow-up set of the solution is the whole domain.

In this paper, we want to know the effect of the singularity, degeneracy and localized reaction on the behavior of the solution of (1.1) and show that the blow-up set of the solution of (1.1) is the whole domain. Our main results are stated as follows.

Theorem 1.1. *Let $(u(x,t), v(x,t))$ be the solution of (1.1), if there exist $a_1, a_2 > 0$, such that $u_0(x) \leqslant a_1\psi(x)$, $v_0(x) \leqslant a_2\varphi(x)$, then $(u(x,t), v(x,t))$ exists globally, where $\psi(x) = \frac{a^{2-r_1}}{2-r_1}\left(\frac{x}{a}\right)^{1-r_1}\left(1-\frac{x}{a}\right)$ and $\varphi(x) = \frac{a^{2-r_2}}{2-r_2}\left(\frac{x}{a}\right)^{1-r_2}\left(1-\frac{x}{a}\right)$.*

Theorem 1.2. *If $u_0(x)$, $v_0(x)$ are sufficiently large in a neighborhood of x_0, then the solution $(u(x,t), v(x,t))$ of problem (1.1) blows up in a finite time.*

Theorem 1.3. *When $q_1 > 0$, $r_1 = 0$ or $q_2 > 0$, $r_2 = 0$, if the solution of problem (1.1) blows up in a finite time, then the blow-up set is $\overline{D}$.*