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# On a multi-scale element-free Galerkin method for the Stokes problem ☆

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ABSTRACT

In this paper, a multi-scale element-free Galerkin method is presented for the Stokes problem. The new method is based on the Hughes' variational multi-scale formulation, and arises from a decomposition of the velocity field into coarse/resolved scales and fine/unresolved scales. In this method, an unresolved model is obtained in which unresolved scales are incorporated analytically through the bubble functions. Modeling of the unresolved scales corrects the lack of the stability of the standard element-free Galerkin formulation and the resulting stabilized formulation possesses superior properties like that of the streamline upwind/Petrov–Galerkin (SUPG) method and the Galerkin/least-squares (GLS) method. The method allows equal order basis for pressure and velocity because it violates the celebrated Babuska–Brezzi condition. A significant feature of the present method is that the structure of the stabilization tensor $\boldsymbol{\tau}$ appears naturally via the solution of the fine-scale problem. Numerical results for example problems confirm that this method has some excellent properties, such as better stability and accuracy.



## 1. Introduction

Nowadays, the methods based on mesh are dominating the other methods for the fluid dynamics problems solved numerically. In order to obtain better numerical results, high-quality mesh which has the ability to capture the fluid flow automatically needs to be generated. However, this process is very difficult to implement, and will consume a lot of time and labor. For the purpose of avoiding mesh generation, a new type of numerical method called meshless or meshfree method [1–6] has been developed in the realm of computational mechanics. So far, there have existed many kinds of meshfree method. Among these, the element-free Galerkin (EFG) method proposed by Belytschko et al. [3–6] possesses a preferable foundation of mathematical theory. When the Stokes problem is solved by the standard EFG method which uses equal order basis for pressure and velocity, we will not be able to obtain the satisfactory numerical results, not to mention the incompressible Navier–Stokes equations. Following the expression of the same phenomenon in the finite element method (FEM), we say that the Babuska–Brezzi condition is not satisfied in the standard EFG method.

Hughes' variational multi-scale method (HVM) [7,8] offers a new perspective for the stabilized methods. The starting point of the method is to decompose the solution into two scales: $\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}'$. Then this multi-scale method tries to determine $\boldsymbol{u}'$ analytically and $\bar{\boldsymbol{u}}$ numerically. Since HVM was proposed, the stabilized methods have made great progress. Hughes [7] showed that the stabilization parameter could be derived from the variational multi-scale formulation. Hauke [9] applied the method to the 1-D advection–reaction and advection–diffusion–reaction problem to obtain an explicit expression for the stabilization parameter. Masud [10,11] developed a multi-scale finite element method for the advection–diffusion equation and the incompressible Navier–Stokes equations. Franca [12–14] proposed a two-level finite element for the

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convection–diffusion problem and the incompressible Navier–Stokes equations and so on [15]. The work of Masud and Franca demonstrates that the numerical method based on HVM will eliminate the explicit dependence of fine scales $\boldsymbol{u}'$, while the ensuing terms will model their effect.

In order to be able to obtain simultaneously the satisfactory numerical results for pressure and velocity by EFG using equal order basis, especially the linear basis that is easy to implement, in this paper we develop a new kind of numerical method called the variational multi-scale element-free Galerkin (VMEFG) method which is also based on the variational multi-scale method [7,8]. Meanwhile, a 2-D advection–diffusion problem, which is a test of very high Peclet number flows and is used to assess solutions which are essentially purely advective in nature, is solved numerically to test the method. Additionally, in order to confirm that the restriction of the Babuska–Brezzi condition can be avoided when this method uses equal order basis for pressure and velocity, the Stokes problem is also solved numerically. So far few papers [16] have been concerned in this topic.

An outline of the paper is as follows: Section 2 presents the fundamental principle of the EFG method. Emphasis in the paper is the description of the VMEFG method, which is presented in Section 3. Section 4 presents the numerical results, and conclusions are drawn in Section 5.

## 2. The fundamental principle of the EFG method

Assume that we have known the nodal value $u_I = u(\boldsymbol{x}_I)$ for the function $u(\boldsymbol{x})$ at $n$ nodes $\boldsymbol{x}_I$ ($I = 1,2,\ldots,n$) in the domain $\Omega$, and a global approximation to the function $u(\boldsymbol{x})$ is given by $u^h(\boldsymbol{x})$. According to the moving least square (MLS) interpolant [3,4], a local approximation $u^h(\boldsymbol{x},\bar{\boldsymbol{x}})$ to the function $u(\boldsymbol{x})$ in the domain $\Omega_{\boldsymbol{x}}$ of influence of node $\boldsymbol{x}$ can be defined by

$$u^h(\boldsymbol{x},\bar{\boldsymbol{x}}) = \sum_{i=1}^{m} p_i(\bar{\boldsymbol{x}})a_i(\boldsymbol{x}) = \boldsymbol{p}^{\mathrm{T}}(\bar{\boldsymbol{x}})\boldsymbol{a}(\boldsymbol{x}), \tag{1}$$

where $m$ is the number of terms in the basis, $p_i(\bar{\boldsymbol{x}})$ are monomial basis functions, and $a_i(\boldsymbol{x})$ are their coefficients, which as indicated, are functions of the spatial coordinates $\bar{\boldsymbol{x}} = (x,y,z)^{\mathrm{T}}$. Examples of commonly used bases are the linear basis:

$$\boldsymbol{p}^{\mathrm{T}} = (1,x) \text{ in 1D} \quad \boldsymbol{p}^{\mathrm{T}} = (1,x,y) \text{ in 2D}$$

and the quadratic basis:

$$\boldsymbol{p}^{\mathrm{T}} = (1,x,x^2) \text{ in 1D} \quad \boldsymbol{p}^{\mathrm{T}} = (1,x,y,x^2,xy,y^2) \text{ in 2D}.$$

Discrete the domain $\Omega$ with $n$ nodes, and define a compact support of isotropic weight function $w_I(\boldsymbol{x}) = w(\boldsymbol{x}-\boldsymbol{x}_I)$ for every node $\boldsymbol{x}_I$ ($I = 1,2,\ldots,n$). Because the weight functions will affect the continuity of the shape functions directly, in this paper we choose cubic spline [2] as the weight functions.

Minimizing the difference between the local approximation $u^h(\boldsymbol{x},\bar{\boldsymbol{x}})$ and the function, this yields the quadratic form:

$$J = \sum_{I=1}^{n} w(\boldsymbol{x}-\boldsymbol{x}_I)\left[\sum_{i=1}^{m} p_i(\boldsymbol{x}_I)a_i(\boldsymbol{x}) - u_I\right]^2. \tag{2}$$

To find the coefficients $\boldsymbol{a}(\boldsymbol{x}) = (a_1(\boldsymbol{x}),a_2(\boldsymbol{x}),\ldots,a_m(\boldsymbol{x}))^{\mathrm{T}}$, we obtain the extremum of $J$ by

$$\frac{\partial J}{\partial \boldsymbol{a}} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{a}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x})\boldsymbol{u} = 0.$$

So we have $\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\boldsymbol{u}$, where $\boldsymbol{A} = \boldsymbol{p}^{\mathrm{T}}\boldsymbol{W}(\boldsymbol{x})\boldsymbol{p}$, $\boldsymbol{B} = \boldsymbol{p}^{\mathrm{T}}\boldsymbol{W}(\boldsymbol{x})$, $\boldsymbol{u} = (u_1,u_2,\ldots,u_n)^{\mathrm{T}}$,

$$\boldsymbol{p} = \begin{bmatrix} p_1(\boldsymbol{x}_1) & p_2(\boldsymbol{x}_1) & \cdots & p_m(\boldsymbol{x}_1) \\ p_1(\boldsymbol{x}_2) & p_2(\boldsymbol{x}_2) & \cdots & p_m(\boldsymbol{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(\boldsymbol{x}_n) & p_2(\boldsymbol{x}_n) & \cdots & p_m(\boldsymbol{x}_n) \end{bmatrix}, \quad \boldsymbol{W}(\boldsymbol{x}) = \begin{bmatrix} w(\boldsymbol{x}-\boldsymbol{x}_1) & 0 & \cdots & 0 \\ 0 & w(\boldsymbol{x}-\boldsymbol{x}_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & w(\boldsymbol{x}-\boldsymbol{x}_n) \end{bmatrix}.$$

Substitute $\boldsymbol{a}(\boldsymbol{x})$ into (1), then the approximation $u^h(\boldsymbol{x},\bar{\boldsymbol{x}})$ can be expressed as

$$u^h(\boldsymbol{x},\bar{\boldsymbol{x}}) = \boldsymbol{p}^{\mathrm{T}}(\bar{\boldsymbol{x}})\boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\boldsymbol{u} = \phi^{\mathrm{T}}(\boldsymbol{x},\bar{\boldsymbol{x}})\boldsymbol{u},$$

where the shape functions of EFG are given by

$$\phi^{\mathrm{T}}(\boldsymbol{x},\bar{\boldsymbol{x}}) = \boldsymbol{p}^{\mathrm{T}}(\bar{\boldsymbol{x}})\boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x}).$$

The above approximation $u^h(\boldsymbol{x},\bar{\boldsymbol{x}})$ is just the local optimal approximation in the domain $\Omega_{\boldsymbol{x}}$ of influence of node $\boldsymbol{x}$ in the sense of MLS. We can deduce a due local optimal approximation $u^h(\boldsymbol{x},\bar{\boldsymbol{x}})$ for the function $u(\boldsymbol{x})$ in which $\boldsymbol{x}$ belongs to all the nodes in $\Omega$. The assemble of all local optimal approximations $u^h(\boldsymbol{x},\bar{\boldsymbol{x}})$ at node $\bar{\boldsymbol{x}} = \boldsymbol{x}$ will form the global approximation $u^h(\boldsymbol{x})$ in $\Omega$, that is to say

$$u(\boldsymbol{x}) \approx u^h(\boldsymbol{x}) = u^h(\boldsymbol{x},\bar{\boldsymbol{x}})|_{\bar{\boldsymbol{x}}=\boldsymbol{x}} = \phi^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\boldsymbol{u}.$$