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On quadrature formulae based on derivative collocation

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Abstract

The recent article "A new type of weighted quadrature rules and its relation to orthogonal polynomials" by Masied-Jamei [M. Masjed-Jamei, A new type of weighted quadrature rules and its relation to orthogonal polynomials, Appl. Math. Comput. 188 (2007) 154–165] introduces quadrature rules based on the evaluation of the derivative(s) of the integrand function rather the function itself. The approach appears useful when a number of derivatives, including the integrand, vanish at a point λ, leading to increased order of accuracy compared to standard Gaussian rules. It is also shown by Masjed-Jamei (2007) how the nodes and weights of the resulting quadrature formula relate to nodes and weights of standard Gaussian quadratures applied to a weight function w to be determined by solving a specific system of integral equations. We give here an explicit expression for w and provide strategies for the practical computation of the quadrature nodes and weights. Additional comments on the examples used by Masjed-Jamei (2007) as well as a generalization involving multiple λ's, are also included.

Keywords: Numerical quadrature; Orthogonal polynomial

1. Introduction

In [5], Masied-Jamei introduces quadrature rules of the form,

$$\int_{\beta}^{\alpha} g(x)\rho(x)\,\mathrm{d}x \approx \sum_{j=1}^{N} \rho_j f^{(m)}(x_j), \tag{1}$$

where

$$g(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(\lambda)\frac{(x-\lambda)^k}{k!} \tag{2}$$

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is the difference between f and its Taylor polynomial of degree (at most) $m-1$ at some value λ. Note that

- $g(\lambda)=\cdots=g^{(m-1)}(\lambda)=0$ and
- (1) is automatically satisfied for $f(x)=1,(x-\lambda),\ldots,(x-\lambda)^{(m-1)}$.

In particular, the quadrature (1) can be expected to be exact for polynomials of degree $2N+m-1$ upon appropriately selecting its nodes and weights. This is not entirely surprising, since (1) can be restated as a rule

$$\int_\alpha^\beta f(x)\rho(x)\mathrm{d}x \approx \sum_{k=0}^{m-1}\tilde{\rho}_k f^{(k)}(\lambda)+\sum_{j=1}^{N}\rho_j f^{(m)}(x_j)$$

with N nodes and $N+m$ weights to be determined. Substituting $f(x)=(x-\lambda)^k$ shows that the weights $\tilde{\rho}_k, k=0,\ldots,m-1$, can be expressed explicitly in terms of the basic moments of order up to $m-1$ at λ:

$$\tilde{\rho}_k \equiv \frac{\mu_k}{k!},\quad \mu_k \equiv \int_\alpha \beta(x-\lambda)^k\rho(x)\mathrm{d}x. \tag{3}$$

It is shown in [5] how the remaining nodes and weights can be obtained from standard Gaussian quadratures on an interval $[a,b]$ (to be determined) by solving integral equations

$$\int_a^b (t-\lambda)^j w(t)\mathrm{d}t=\frac{j!}{(j+m)!}\int_\alpha^\beta (x-\lambda)^{j+m}\rho(x)\mathrm{d}x,\quad j=0,\ldots,2N-1 \tag{4}$$

for a weight function w (compare [5, Eq. (25)]). Provided $[a,b]$ and w can be found such that (4) holds, the nodes and weights of the quadrature (1) are then simply given by the nodes $x_j=t_j$ and weights ρ_j of the Gaussian quadrature,

$$\int_a^b f(t)w(t)\mathrm{d}t \approx \sum_{j=1}^{N}\rho_j f(t_j). \tag{5}$$

In this note, we show that a weight function w can in fact be explicitly determined in terms of ρ, using the Peano kernel representation of the function g itself (in [5] Masjed-Jamei uses such representation in the error analysis, but missed the connection in the derivation of the rule, despite coming close in [5, Section 2.1], when commenting on possible applications). We show that w is definite when λ in outside the (open) interval of integration (α,β) (Section 2) but indefinite (changes sign) when $\alpha<\lambda<\beta$ and m is odd (Section 3). We nevertheless show that the only example of this type considered in [5] can be reformulated, because of symmetry, as a rule (1) for which $\lambda=\alpha$. In Section 4 we comment on the practical determination of the nodes and weights, which is not addressed in [5]. Finally, additional issues relevant to the quadrature (1) are included in Section 5. In particular, a generalization of (1), where g vanishes at m distinct points is presented.

2. Definite case: $\lambda \notin (\alpha,\beta)$

W.l.o.g. assume $\lambda \leqslant \alpha$ (otherwise let $x\to -x$ in (1)). The Peano kernel representation of the linear functional $f\to g$ yields

$$g(x)=\int_\lambda^x \frac{(x-t)^{m-1}}{(m-1)!}f^{(m)}(t)\mathrm{d}t, \tag{6}$$

which can be shown via successive integration by parts. Therefore, the integral on the left-hand side of (1) can be written as