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Two-step almost collocation methods for Volterra integral equations

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ABSTRACT

In this paper we construct a new class of continuous methods for Volterra integral equations. These methods are obtained by using a collocation technique and by relaxing some of the collocation conditions in order to obtain good stability properties.

1. Introduction

In this paper we analyze the construction of highly stable continuous methods for the numerical solution of Volterra integral equations (VIEs) of the second kind

$$y(t) = g(t) + \int_{t_0}^{t} k(t,\eta,y(\eta))\mathrm{d}\eta, \quad t \in [t_0, T], \tag{1.1}$$

where the functions $g : \mathbb{R} \to \mathbb{R}^D$ and $k : \mathbb{R}^2 \times \mathbb{R}^D \to \mathbb{R}^D$ are assumed to be sufficiently smooth. Let N be a positive integer and consider the uniform grid

$$t_n = t_0 + nh, \quad n = 0, 1, \ldots, N, \quad Nh = T - t_0.$$

To define numerical methods for (1.1) it is convenient to split for $t \in [t_n, t_{n+1}]$ the integral appearing in this equation into two parts, and to rewrite the equation in the form

$$y(t) = F^{[n]}(t) + \Phi^{[n+1]}(t) \tag{1.2}$$

with the lag-term $F^{[n]}(t)$ defined by

$$F^{[n]}(t) = g(t) + \int_{t_0}^{t_n} k(t,\eta,y(\eta))\mathrm{d}\eta$$

and the increment term $\Phi^{[n+1]}(t)$ defined by

$$\Phi^{[n+1]}(t) = \int_{t_n}^{t} k(t,\eta,y(\eta))\mathrm{d}\eta.$$

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Observe that we have supressed the dependence of $F^{[n]}(t)$ and $\Phi^{[n+1]}(t)$ on $y(t)$. We will look for a continuos approximation $P(t_n+sh)$, $s\in[0,1]$, to the solution $y(t_n+sh)$ of (1.1), which employs the information about the equation on two consecutive steps. The methods for (1.1) are defined by

$$\begin{cases} P(t_n+sh)=\varphi_0(s)y_{n-1}+\varphi_1(s)y_n+\sum\limits_{j=1}^{m}\chi_j(s)P(t_{n-1,j})+\sum\limits_{j=1}^{m}\psi_j(s)(F_h^{[n]}(t_{n,j})+\Phi_h^{[n+1]}(t_{n,j})),\\ y_{n+1}=P(t_{n+1}), \end{cases} \tag{1.3}$$

where $s\in(0,1]$, $n=1,2,\ldots,N-1$. Here, $c=[c_1,\ldots,c_m]^{\mathrm{T}}$ is the abscissa vector, $t_{n-1,j}=t_{n-1}+c_jh$, $t_{n,j}=t_n+c_jh$, and $F_h^{[n]}(t)$, $\Phi_h^{[n+1]}(t)$ are approximations to $F^{[n]}(t)$, $\Phi^{[n+1]}(t)$ which are usually computed by appropriate quadrature formulas of sufficiently high order. The polynomial $P(t_n+sh)$ will be explicitly defined after solving, at each step, a system of nonlinear equations in the stage values $Y_i^{[n+1]}=P(t_{n,i})$ and y_{n+1}. The polynomials $\varphi_0(s)$, $\varphi_1(s)$, $\chi_j(s)$ and $\psi_j(s)$, $j=1,2,\ldots,m$, which define the method (1.3) will be determined from the continuous order conditions derived in Section 2.

It will be assumed that the approximations $F_h^{[n]}(t)$, $\Phi_h^{[n+1]}(t)$ to $F^{[n]}(t)$, $\Phi^{[n+1]}(t)$ take the following forms:

$$F_h^{[n]}(t)=g(t)+h\sum_{v=1}^{n}\left(b_0k(t,t_{v-1},y_{v-1})+\sum_{j=1}^{m}b_jk(t,t_{v-1,j},P(t_{v-1,j}))+b_{m+1}k(t,t_v,y_v)\right) \tag{1.4}$$

and

$$\Phi_h^{[n+1]}(t_{n,i})=h\left(w_{i,0}k(t_{n,i},t_n,y_n)+\sum_{j=1}^{m}w_{i,j}k(t_{n,i},t_{n,j},P(t_{n,j}))+w_{i,m+1}k(t_{n,i},t_{n+1}y_{n+1})\right), \tag{1.5}$$

with given weights b_0, b_j, b_{m+1}, $w_{i,0}$, w_{ij} and $w_{i,m+1}$. Observe that these formulas employ the values of the kernel k at the points t_v and $t_{v,j}$.

Computing (1.3) for $s=c_i$, $i=1,2,\ldots,m$, and $s=1$, we obtain

$$\begin{cases} Y_i^{[n+1]}=\varphi_0(c_i)y_{n-1}+\varphi_1(c_i)y_n+\sum\limits_{j=1}^{m}\chi_j(c_i)Y_j^{[n]}+\sum\limits_{j=1}^{m}\psi_j(c_i)(F_h^{[n]}(t_{n,j})+\Phi_h^{[n+1]}(t_{n,j})),\\ y_{n+1}=\varphi_0(1)y_{n-1}+\varphi_1(1)y_n+\sum\limits_{j=1}^{m}\chi_j(1)Y_j^{[n]}+\sum\limits_{j=1}^{m}\psi_j(1)(F_h^{[n]}(t_{n,j})+\Phi_h^{[n+1]}(t_{n,j})), \end{cases} \tag{1.6}$$

$n=1,2,\ldots,N-1$, where $Y_i^{[n+1]}=P(t_{n,i})$, $Y_i^{[n]}=P(t_{n-1,i})$. Introducing the notation

$$Y^{[n]}=\begin{bmatrix}Y_1^{[n]}\\ \vdots\\ Y_m^{[n]}\end{bmatrix},\quad F_h^{[n]}(t_{n,c})=\begin{bmatrix}F_h^{[n]}(t_{n,1})\\ \vdots\\ F_h^{[n]}(t_{n,m})\end{bmatrix},\quad \Phi_h^{[n+1]}(t_{n,c})=\begin{bmatrix}\Phi_h^{[n+1]}(t_{n,1})\\ \vdots\\ \Phi_h^{[n+1]}(t_{n,m})\end{bmatrix},\quad \varphi_0(c)=\begin{bmatrix}\varphi_0(c_1)\\ \vdots\\ \varphi_0(c_m)\end{bmatrix},\quad \varphi_1(c)=\begin{bmatrix}\varphi_1(c_1)\\ \vdots\\ \varphi_1(c_m)\end{bmatrix},$$

$$\chi(1)=\begin{bmatrix}\chi_1(1)\\ \vdots\\ \chi_m(1)\end{bmatrix},\quad \psi(1)=\begin{bmatrix}\psi_1(1)\\ \vdots\\ \psi_m(1)\end{bmatrix},$$

and

$$A=[\chi_j(c_i)]_{i,j=1}^{m},\quad B=[\psi_j(c_i)]_{i,j=1}^{m},$$

the formula (1.6) can be rewritten in the following vector form:

$$\begin{cases} Y^{[n+1]}=(\varphi_0(c)\otimes I)y_{n-1}+(\varphi_1(c)\otimes I)y_n+(A\otimes I)Y^{[n]}+(B\otimes I)(F_h^{[n]}(t_{n,c})+\Phi_h^{[n+1]}(t_{n,c})),\\ y_{n+1}=\varphi_0(1)y_{n-1}+\varphi_1(1)y_n+(\chi^{\mathrm{T}}(1)\otimes I)Y^{[n]}+(\psi^{\mathrm{T}}(1)\otimes I)(F_h^{[n]}(t_{n,c})+\Phi_h^{[n+1]}(t_{n,c})), \end{cases} \tag{1.7}$$

$n=1,2,\ldots,N-1$, where I is the identity matrix of dimension D. The polynomial $P(t)$ in (1.3) provides a continuous approximation of the solution on the interval of integration $[t_0,T]$.

The idea of two-step Runge–Kutta and collocation methods has been widely developed in the context of ordinary differential equations (ODEs) and delay differential equations (DDEs). In particular the papers [1,3,11,12,14,16,18,22,23] and the monograph [15] are concerned with discrete two-step Runge–Kutta methods for ODEs, while the papers [4,6,13,17,2,5] deal with continuous two-step collocation and two-step Runge–Kutta methods for ODEs and DDEs, respectively.

In this paper we will construct highly stable almost two-step collocation methods for VIEs, following the approach proposed in [13] in the context of ODEs. This technique is based on using a collocation technique and then relaxing some of the collocation and interpolation conditions in order to obtain desirable stability properties. We will analyze the convergence and order of the proposed methods and provide the stability analysis with respect to the basic test equation