Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics and Computation 200 (2008) 547–556
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Existence of weakly efficient solutions in nonsmooth vector optimization

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Abstract

In this paper we study the existence of weakly efficient solutions for some nonsmooth and nonconvex vector optimization problems. We consider problems whose objective functions are defined between infinite and finite-dimensional Banach spaces. Our results are stated under hypotheses of generalized convexity and make use of variational-like inequalities.

Keywords: Multiobjective optimization; Nonsmooth analysis; Invexity; Variational-like inequalities

1. Introduction

The connection between variational inequalities and optimization problems is well known (e.g. [1,6,11,12]) and have been extensively investigated in the recent years by several authors. One of the main works in this direction was done by Giannessi [8], where many existence results for optimization problems were obtained by using variational inequalities.

For multiobjective optimization problems, Giannessi proved in [9] that there exists an equivalence between efficient solutions of differentiable convex optimization problems and the solutions of a variational inequality of Minty type. He also established similar results for efficient solutions. On the other hand, using subdifferentials, Lee showed in [13] that analogous results are true for nonsmooth convex problems defined between finite-dimensional spaces.

For some nonconvex differentiable vector problems defined between infinite-dimensional Banach spaces, Chen and Craven [4] proved the equivalence of weakly efficient solutions and the solutions of a certain

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2007.11.027

variational-like inequality. Using this characterization they proved an existence result for weakly efficient solutions.

In this work we consider the following two problems.

1. An infinite-dimensional problem:

$$\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & x \in K, \end{array} \tag{P1}$$

where X and Y are two Banach spaces, $f : X \to Y$ is a given function and K is a nonempty subset of X.

2. A finite-dimensional problem:

$$\begin{array}{ll} \text{Minimize} & f(x) := (f_1(x), \ldots, f_p(x)) \\ \text{subject to} & x \in X, \end{array} \tag{P2}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, p)$ are given functions and X is a nonempty subset of $\mathbb{R}^n$.

For both problems, by "minimize" we mean "find the weakly efficient solution of the problem" Our objective is to solve problem (P1) without assuming hypotheses of differentiability, which extends early results by Chen and Craven [4], and to solve problem (P2) under assumptions of generalized convexity, which extends early results by Lee [13].

This paper is organized as follows: In Section 2 we fix the notation and recall some facts from nonsmooth analysis. In Section 3 we consider the problem (P1) and establish our existence result. In Section 4 we consider the problem (P2).

2. Preliminaries

Let X and Y be two real Banach spaces. We will denote by $\|\cdot\|$ the norm in Y. Let K be a nonempty subset of X and $P \subset Y$ a pointed convex cone (i.e. $P \bigcap (-P) = \{0\}$) such that $\operatorname{int} P \neq \emptyset$. Also, let $f : X \to Y$ be a given function. We consider the problem (P1) given in the previous section. The notion of optimality (or *equilibria*) that we consider here is the *weak efficiency*. We say that $x_0 \in K$ is a *weakly efficient solution* of (P1) if

$$f(x) - f(x_0) \notin -\operatorname{int} P, \quad \forall x \in K.$$

In particular, for the problem (P2), the definition of weakly efficient solution is done by taking $P = \mathbb{R}^p_+$ in the previous definition, that is, $x_0 \in X$ is a weakly efficient solution of (P2) if does not exist $x \in X$ such that

$$f_i(x) < f_i(x_0), \quad \forall i = 1, \ldots, p.$$

Now, we recall some notions and results from nonsmooth analysis. Let ϕ be a locally Lipschitz function from a Banach space X into $\mathbb{R}$. The *Clarke generalized directional derivative* of ϕ, at a point $\bar{x} \in X$, and in the direction $d \in X$, denoted by $\phi^0(\bar{x}; d)$, is given by:

$$\phi^0(\bar{x}; d) = \limsup_{\substack{y \to \bar{x} \\ t \downarrow 0}} \frac{\phi(y + td) - \phi(y)}{t}$$

and the *Clarke generalized gradient* of ϕ at $\bar{x}$ is given by

$$\partial\phi(\bar{x}) = \{x^* \in X^* : \phi^0(\bar{x}; d) \geqslant \langle x^*, d\rangle, \ \forall d \in X\},$$

where X^* denotes the topological dual of X and $\langle \cdot, \cdot \rangle$ is the canonical bilinear form pairing X^* and X.

The next proposition establish some properties of the generalized directional derivative and the generalized gradient of Clarke.

Proposition 1. *Let $f : \Omega \to \mathbb{R}$ be a locally Lipschitz function with Lipschitz constant k. Then the following assertions are true*: